ELT2035 Signals & Systems

Lesson 11: Laplace transform exercises

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Exercise 1

Consider the signal $x(t) = 3e^{2t}u(t) + 4e^{3t}u(t)$.

a. Does the Fourier transform of this signal converge?

b. For which of the following values of a does the Fourier transform of $x(t)e^{-\sigma t}$ converge? (i) $\sigma = 1$ (ii) $\sigma = 2.5$ (iii) $\sigma = 3.5$

c. Determine the Laplace transform of x(t). Sketch the location of the poles and zeros of X(s) and the ROC.

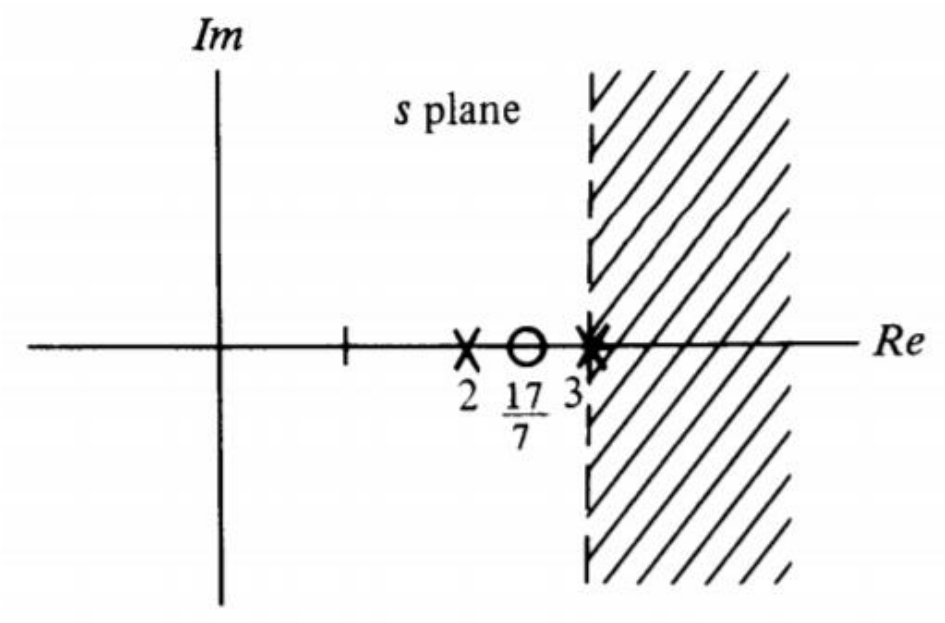
SOLUTION

a. The Fourier transform of the signal does not exist because of the presence of growing exponentials. In other words, $x(t)$ is not absolutely integrable.

b. (iii) $\sigma = 3.5$, for any value of a > 3, the signal $x(t)e^{-\sigma t}$ decays exponentially, and the Fourier transform converges.

c. The Laplace transform of $x(t)$ is $X(s) = \frac{3}{s-2} + \frac{4}{s-3} = \frac{7s-17}{(s+2)(s+3)}$, and its pole-zero plot and ROC are as shown in the below figure:

Exercise 1 solution (cont.)

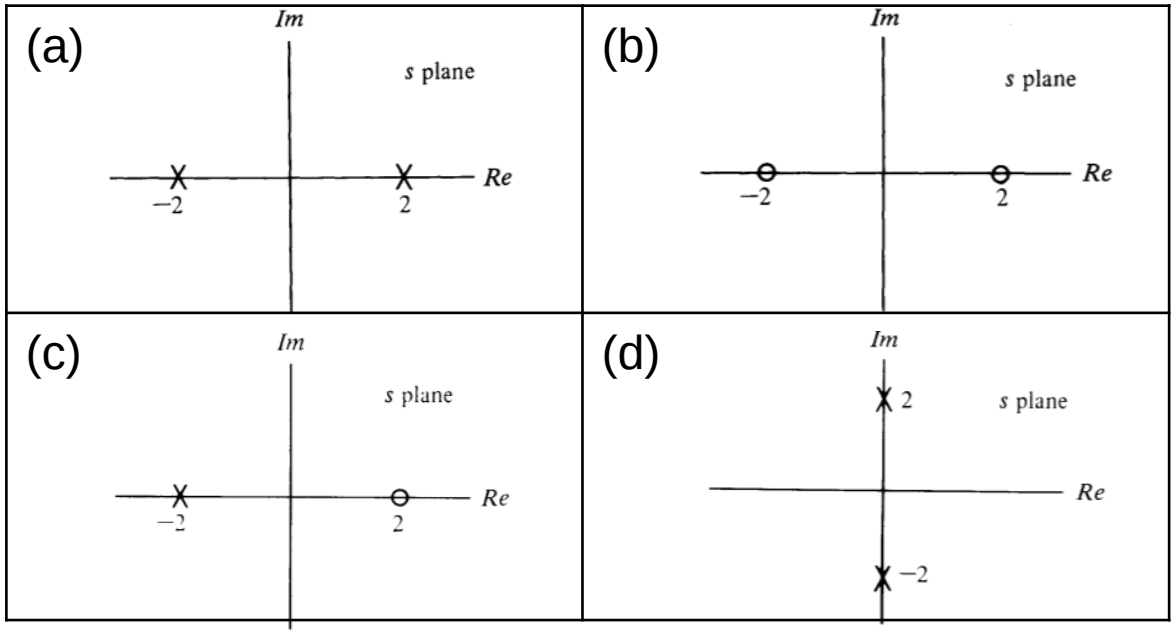


Exercise 2

Fill in the corresponding constraint on the ROC for the time function *x(t)* associated with each pole-zero plot below:

Constraint on ROC for Pole-Zero Pattern

$x(t)$
(i) Fourier transform of $x(t)e^{-t}$ converges
(ii) $x(t) = 0$, $t > 10$
(iii) $x(t) = 0$, $t < 0$

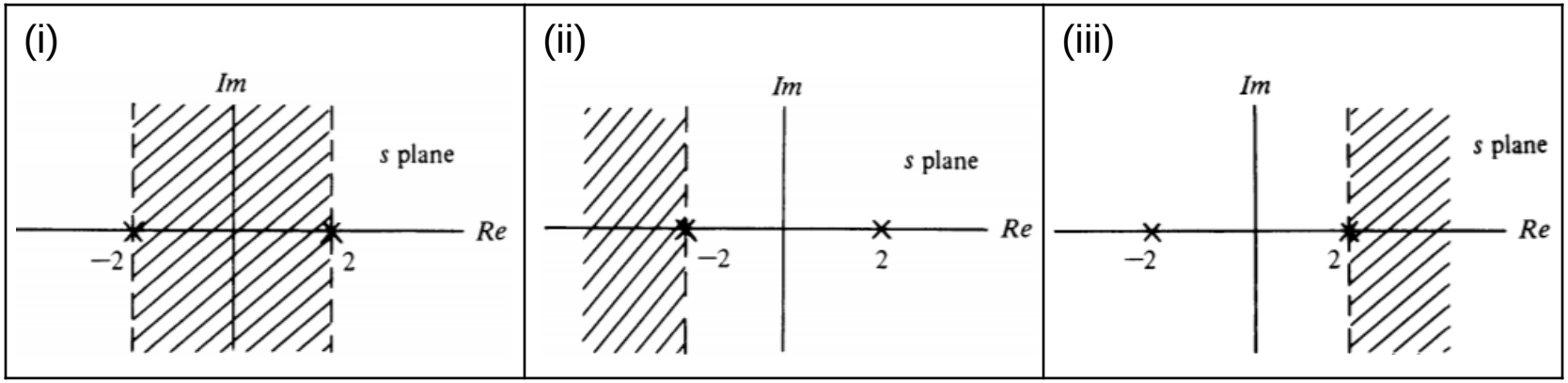


Exercise 2 solution

a. (i) Since the Fourier transform of $x(t)e^{-t}$ exists, $\sigma = 1$ must be in the ROC $\rightarrow$ there's only one possible ROC: $-2 < \sigma < 2$.

(ii) We are specifying a left-sided signal $\rightarrow$ the ROC is: $\sigma < -2$.

(iii) We are specifying a right-sided signal $\rightarrow$ the ROC is: $2 < \sigma$.



b. Since there are no poles present, the ROC exists everywhere in the *s*-plane.

c. (i) $\sigma = 1$ must be in the ROC $\rightarrow$ the ROC is: $\sigma > -2$.

(ii) We are specifying a left-sided signal $\rightarrow$ the ROC is: $\sigma < -2$.

(iii) We are specifying a right-sided signal $\rightarrow$ the ROC is: $-2 < \sigma$.

d. (i) $\sigma = 1$ must be in the ROC $\rightarrow$ the ROC is: $\sigma > 0$.

(ii) We are specifying a left-sided signal $\rightarrow$ the ROC is: $\sigma < 0$.

(iii) We are specifying a right-sided signal $\rightarrow$ the ROC is: $0 < \sigma$.

Exercise 2 solution (cont.)

$x(t)$	Constraint on ROC for Pole-Zero Pattern (a)	(b)	(c)	(d)
(i) Fourier transform of $x(t)e^{-t}$ converges	$-2 < \sigma < 2$	Entire s plane	$\sigma > -2$	$\sigma > 0$
(ii) $x(t) = 0$, $t > 10$	$\sigma < -2$	Entire s plane	$\sigma < -2$	$\sigma < 0$
(iii) $x(t) = 0$, $t < 0$	$\sigma > 2$	Entire s plane	$\sigma > -2$	$\sigma > 0$

Exercise 3

Determine the Laplace transform and associated ROC for each of the following time functions.

a. $x(t) = \sum_{k=0}^{\infty} a^k \delta(t - kT),\ a > 0$

b. $x(t) = \sin(\omega t + b)\, e^{-at} u(t),\ a > 0$

SOLUTION

a. We have:

$$\begin{aligned} X(s) &= \sum_{k=0}^{\infty} a^k \int_{-\infty}^{\infty} \delta(t - kT) e^{-st} dt \\ &= \sum_{k=0}^{\infty} a^k e^{-skT} = \frac{1}{1 - ae^{-sT}} \end{aligned}$$

The ROC is $|ae^{-sT}| < 1 \rightarrow a^2 e^{-2sT} < 0 \rightarrow 2\ln a - 2sT < 1 \rightarrow s > \frac{\ln a}{T}$.

b. Consider $y(t) = \sin(\omega t + b)\, u(t) = (\sin \omega t \cos b + \cos \omega t \sin b) u(t)$, applying linearity property of the LT yields $Y(s) = \cos b \frac{\omega}{s^2 + \omega^2} + \sin b \frac{s}{s^2 + \omega^2}$

Applying frequency shift property to $y(t)$ yields:

$$Y(s) = X(s + a) = \sin b \frac{s + a + \omega \cot b}{(s + a)^2 + \omega^2}.$$

The ROC is $\mathrm{Re}(s) = \sigma > -a$.

Exercise 4

Using the identity $e^{j\omega t} \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s-j\omega}$ and properties of Laplace transform, determine the time function $x(t)$ for each Laplace transform $X(s)$

a. $X(s) = \frac{s}{s^2+4}$, $\mathrm{Re}(s) > 0$.

b. $X(s) = \frac{s+1}{(s+1)^2+4}$, $\mathrm{Re}(s) > -1$.

SOLUTION:

a. We have $X(s) = \frac{s}{s^2+4} = \frac{1}{2}\left(\frac{1}{s+j2} + \frac{1}{s-j2}\right)$, thus $x(t) = \frac{1}{2}\left(e^{j2t} + e^{-j2t}\right)u(t) = \cos(2t)\,u(t)$.

b. Using the result in part a and applying frequency shift property, we obtain $x(t) = e^{-t}\cos(2t)\,u(t)$.

Exercise 5

Determine $x(t)$ for the following conditions if $X(s)$ is given by $X(s) = \frac{1}{(s+1)(s+2)}$.

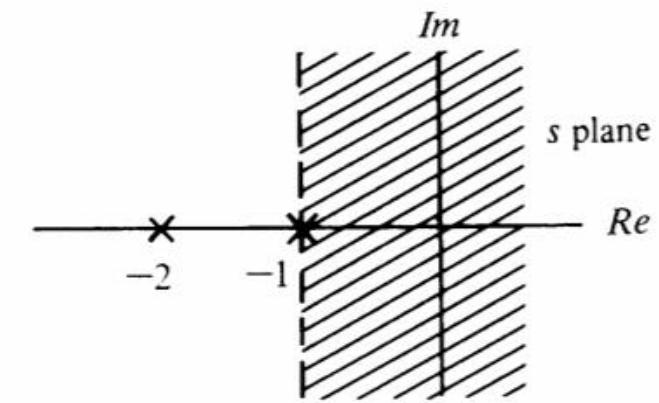
a. $x(t)$ is right-sided

b. $x(t)$ is left-sided

c. $x(t)$ is two-sided

SOLUTION

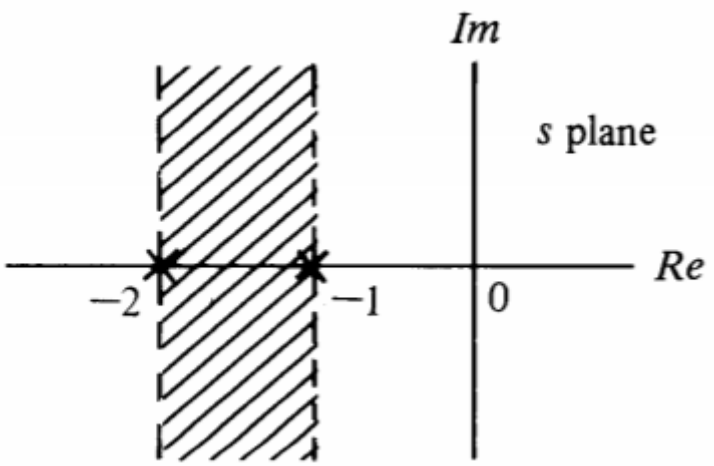
a. We have $X(s) = \frac{1}{x+1} - \frac{1}{x+2}$, thus for $x(t)$ is right-sided the ROC is to the right of the rightmost pole, and hence $x(t) = e^{-t}u(t) - e^{-2t}u(t)$.



b. By similar argument to part a, $x(t) = -e^{-t}u(-t) + e^{-2t}u(-t)$.

Exercise 5 solution (cont.)

c. For the two-sided assumption, we know that $x(t)$ will have the form $x(t) = x_1(t)u(-t) + x_2(t)u(t)$ and the ROC must be the strip between the two poles. Since $\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] = \begin{cases} e^{-t}u(t), & \sigma > -1 \\ -e^{-t}u(-t), & \sigma < -1 \end{cases}$ and $\mathcal{L}^{-1}\left[\frac{1}{s+2}\right] = \begin{cases} e^{-2t}u(t), & \sigma > -2 \\ -e^{-2t}u(-t), & \sigma < -2 \end{cases}$, there's only one solution $x(t) = -e^{-t}u(-t) - e^{-2t}u(t)$ with the ROC depicted below.



Exercise 6

Apply the partial-fraction expansion method to determine $x(t)$ for each of the following Laplace transform

a. $X(s) = \frac{2s+14}{s^2+6s+25}$

b. $X(s) = \frac{4s^2-15s-10}{(s+2)^3}$

SOLUTION:

a. We have $X(s) = \frac{A}{s+3-j4} + \frac{B}{s+3+j4}$ with $A = X(s)(s+3-j4)|_{s=-3+j4} = 1-j$, $B = X(s)(s+3+j4)|_{s=-3-j4} = 1+j$. Hence, $x(t) = \left[(1-j)e^{-(3+j4)t} + (1+j)e^{-(3-j4)t}\right]u(t) = 2\sqrt{2}e^{-3t}\cos(4t-45°)\,u(t)$.

b. We have $X(s) = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{(s+2)^3}$ with $C = (s+2)^3X(s)|_{s=-2} = 36$, $B = \frac{d}{ds}[(s+2)^3X(s)]\Big|_{s=-2} = -31$, $B = \frac{d^2}{ds^2}[(s+2)^3X(s)]\Big|_{s=-2} = 4$. Hence, $x(t) = (4-31t+18t^2)e^{-2t}u(t)$.

Exercise 7

Determine the initial and final values of *x(t)* if its Laplace transform is given by:

a. $X(s) = \frac{s+2}{s^2+2s-3}$

b. $X(s) = e^{-3s}\frac{2s^2+1}{s(s+2)^2}$

SOLUTION:

a. We have

$$x(0^+) = \lim_{s\to\infty} sX(s) = \lim_{s\to\infty}\frac{s^2+2s}{s^2+2s-3} = 1$$

$$x(\infty) = \lim_{s\to 0} sX(s) = \lim_{s\to 0}\frac{s^2+2s}{s^2+2s-3} = 0$$

b. We have

$$x(0^+) = \lim_{s\to\infty} sX(s) = \lim_{s\to\infty} e^{-3s}\frac{2s^2+1}{s^2+2s+4} = 0$$

$$x(\infty) = \lim_{s\to 0} sX(s) = \lim_{s\to 0} e^{-3s}\frac{2s^2+1}{s^2+2s+4} = \frac{1}{4}$$

Exercise 8

An LTI system has impulse response $h(t) = 3e^{-t}u(t) - 2e^{-2t}u(t)$. Determine the differential equation description of the system.

SOLUTION:

We have $H(s) = \mathcal{L}[h(t)] = \frac{3}{s+1} - \frac{2}{s+2} = \frac{s+4}{s^2+3s+2}$.

On the other hand, $H(s) = \frac{Y(s)}{X(s)} \Longrightarrow Y(s)(s^2 + 3s + 2) = X(s)(s + 4)$. (1)

Finally, taking the inverse Laplace transform on both sides of (1) and apply differentiation and linearity properties, assuming zero initial conditions, to yield:

$$\frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) + 2y(t) = \frac{d}{dt}x(t) + 4x(t).$$

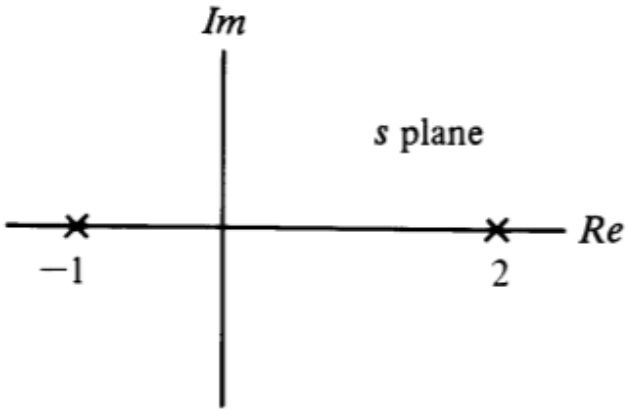
Exercise 9

Consider a continuous-time LTI system for which the input $x(t)$ and output $y(t)$ are related by the differential equation $\frac{d^2}{dt^2}y(t) - \frac{d}{dt}y(t) - 2y(t) = x(t)$.

a. Determine the transfer function $H(s)$ of the system.
b. Sketch the ROC for each of the following cases: (i) The system is stable (ii) The system is causal (iii) The system is neither stable nor causal.
c. Determine $h(t)$ when the system is causal.

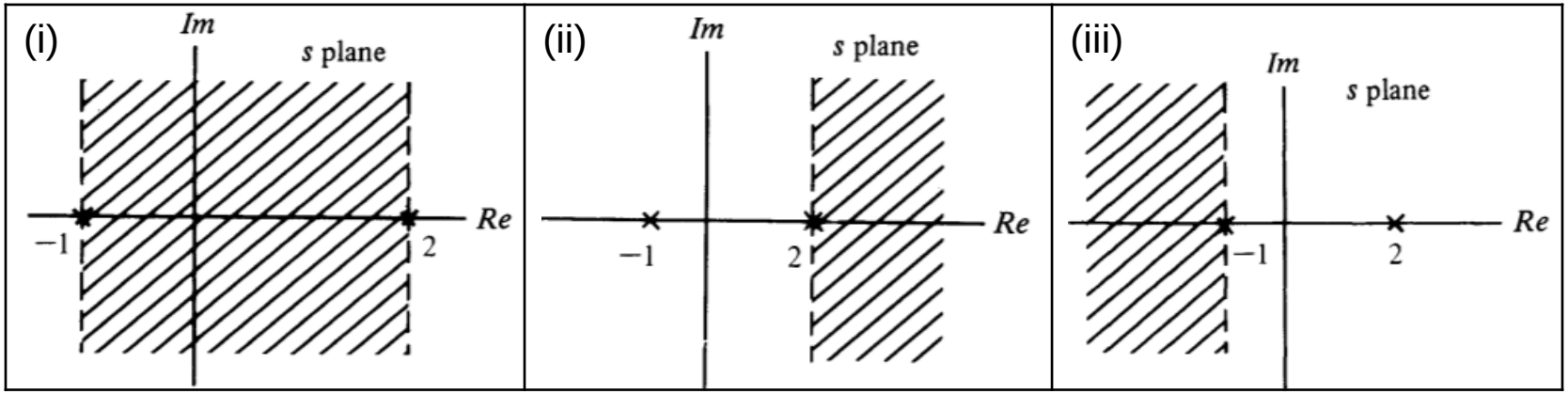
SOLUTION:

a. Take Laplace transform on both sides of the ODE, assuming zero initial conditions, we have $s^2Y(s) - sY(s) - 2Y(s) = X(s)$. Hence $H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2-s-2} = \frac{1}{(s-2)(s+1)}$. The pole-zero plot is shown below



Exercise 9 solution (cont.)

b. (i) For a stable system, the ROC must include the $j\omega$-axis (ii) For a causal system, the ROC must be to the right of the rightmost pole (iii) For a system that is not causal or stable, we are left with an ROC that is to the left of $s = -1$.



c. Since $h(t) = \mathcal{L}^{-1}[H(s)]$, to find the impulse response, we use the partial fraction expansion $H(s) = \frac{-1/3}{s+1} + \frac{1/3}{s-2}$. We now take the inverse Laplace transform of each term in the partial fraction expansion. Since the system is causal, we choose right-sided signals in both cases. Thus,

$$h(t) = -\frac{1}{3}e^{-t}u(t) + \frac{1}{3}e^{2t}u(t).$$