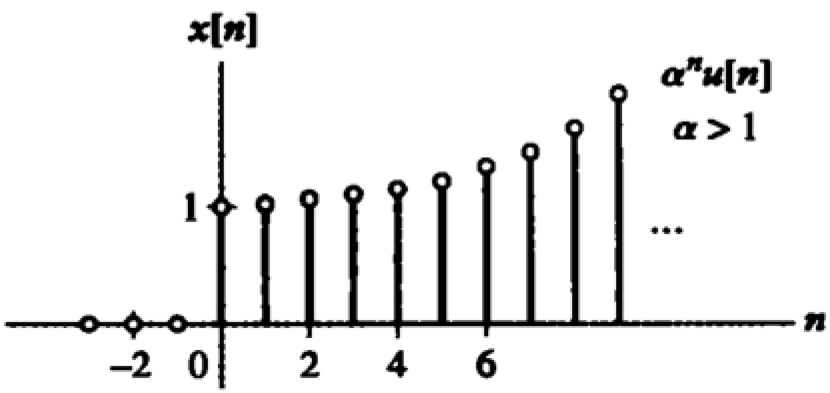
ELT2035 Signals & Systems

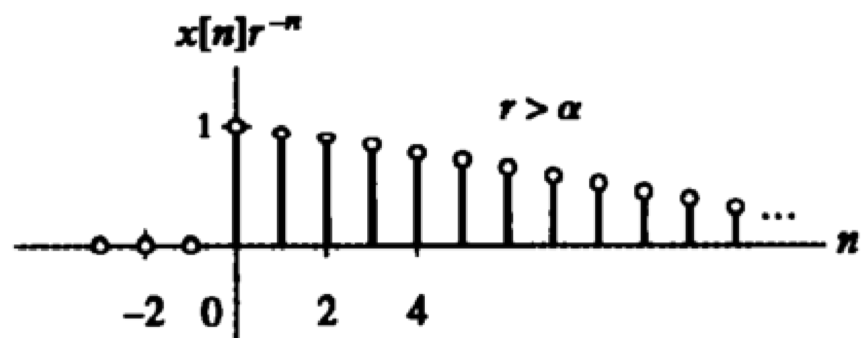
Lesson 12: The z-Transform

Hoang Gia Hung
Faculty of Electronics and Telecommunications
University of Engineering and Technology, VNU Hanoi

Introduction

- Fourier representations of DT signals and LTI systems are based on superpositions of complex sinusoids
  - The DTFT does not exist for signals that are not absolutely summable;
  - The DTFT cannot be ***easily*** used to analyze unstable or even marginally stable systems.
- ***z-transform*** provides a broader characterization of ***discrete*** time signals and LTI systems than Fourier methods
  - Based on superpositions of continuous time complex exponentials of the form $e^{(\sigma+j\Omega)k}$ rather than complex sinusoids $e^{j\Omega k}$.
  - The z-transform exists for signals that do not have a DTFT by selecting a proper value for σ.





The z-transform

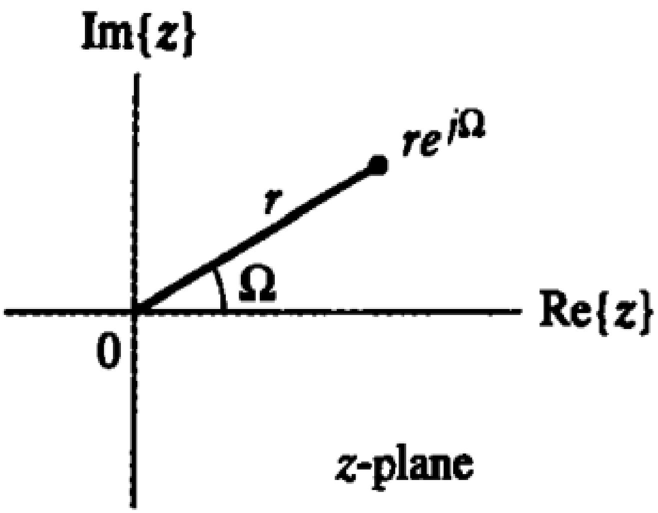
- Consider the Fourier transform of $f(t)e^{-\sigma t}$ (σ real)
  - $\mathcal{F}\{f[k]e^{-\sigma k}\} = \sum_{k=-\infty}^{\infty} f[k]e^{-\sigma k}e^{-j\Omega k} = \sum_{k=-\infty}^{\infty} f[k]\, e^{-(\sigma+j\Omega)k} = F(e^{\sigma+j\Omega})$
  - The inverse DTFT of $F(e^{\sigma+j\Omega})$ is $f[k]e^{-\sigma k} = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{\sigma+j\Omega})e^{j\Omega k}d\Omega$. Multiplying both sides with $e^{\sigma k}$ yields $f[k] = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{\sigma+j\Omega})e^{(\sigma+j\Omega)k}d\Omega$.
  - Changing variable Ω to $z = e^{\sigma+j\Omega} = re^{j\Omega}$ with $r = e^{\sigma}$ yields bilateral z-transform pair, notice that $\ln z = \sigma + j\Omega$ and $\frac{1}{z}dz = jd\Omega$.
- Bilateral z-transform pair

$$F(z) = \sum_{k=-\infty}^{\infty} f[k]z^{-k} \qquad (1)$$

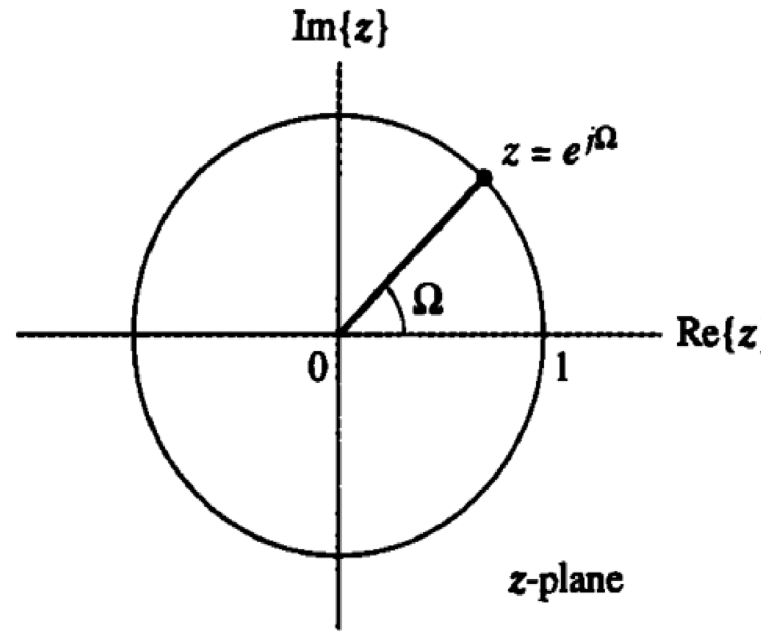
$$f[k] = \frac{1}{2\pi j}\oint F(z)z^{k-1}dz \qquad (2)$$

  - As Ω varies from $-\pi$ to π, z completes exactly one rotation in counterclockwise direction on a circle of radius r $\rightarrow$ the integral in Eq. (2) is a contour integral around a circle of radius r in counterclockwise direction.
  - In practice, we usually do not evaluate (2) directly but use a z-transform table.
  - If we let $\sigma = 0$, $F(z) = F(e^{j\Omega})$$\rightarrow$ the DTFT is a special case of the z-transform obtained by letting $z = e^{j\Omega}$ (i.e. z circumnavigates along the ***unit circle*** on the z-plane).

The z-plane illustration



$z = e^{\sigma + j\Omega} = re^{j\Omega}$



$X\left(e^{j\Omega}\right) = X(z)\big|_{z=e^{j\Omega}}$

The z-transform and the DTFT

Example 7.1 The z-Transform and the DTFT Determine the z-transform of the signal

$$x[n] = \begin{cases} 1, & n = -1 \\ 2, & n = 0 \\ -1, & n = 1 \\ 1, & n = 2 \\ 0, & \text{otherwise} \end{cases}.$$

Use the z-transform to determine the DTFT of $x[n]$.

SOLUTION

- $X(z) = \sum_{k=-\infty}^{\infty} x[k] z^{-k} = z + 2 - z^{-1} + z^{-2}$
- Substituting $z = e^{j\Omega}$ to $X(z)$ yields $X(e^{j\Omega}) = e^{j\Omega} + 2 - e^{-j\Omega} + e^{-2j\Omega}$

Region of convergence (ROC)

- The ROC for *F(z)* is a set of values of *z* (a region in the z-plane) for which the infinite sum in Eq. (1) converges.
  - A necessary condition for convergence is absolute summability of $x[n]z^{-n}$. Since $|x[n]z^{-n}| = |x[n]r^{-n}|$, we must have $\sum_{n=-\infty}^{\infty}|x[n]r^{-n}| < \infty$.
  - For bilateral z-transform, it is possible that two different signals have the same *F(z)* but with different ROC. In other words, there is no 1-to-1 correspondence between *F(z)* and *f[k]* unless the ROC is specified.
  - **Example**: Find the z-transform and ROC for $x[k] = \alpha^k u[k]$
  - **Solution:**

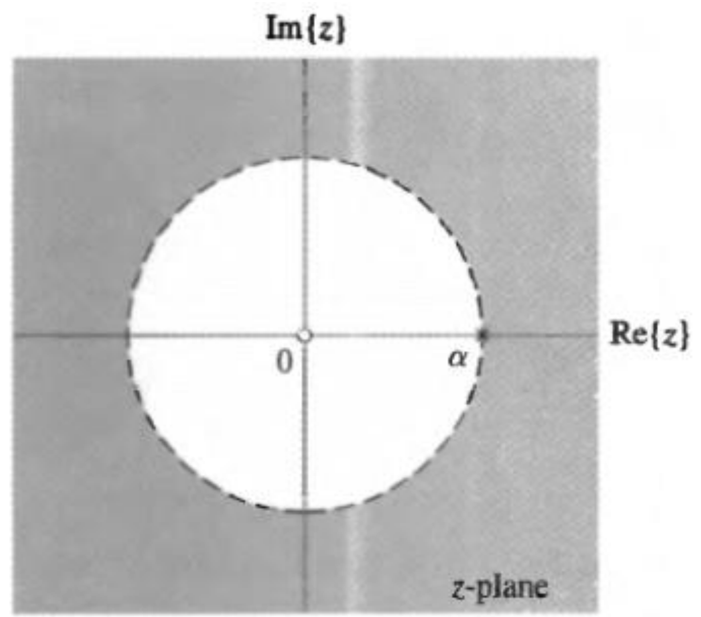
$$X(z) = \sum_{k=-\infty}^{\infty} \alpha^k u[k] z^{-k} = \sum_{k=0}^{\infty} \left(\frac{\alpha}{z}\right)^k$$

- For $|z| > |\alpha|$, the sum converges to

$$X(z) = \frac{1}{1-\frac{\alpha}{z}} = \frac{z}{z-\alpha}.$$

- For $|z| \leq |\alpha|$, the sum does not converge.

Hence, the ROC of $X(z)$ is the shaded region outside the circle of radius $|\alpha|$, centred at the origin in the z-plane.



Poles and zeros

- Like in the CT case, it is useful to express $X(z) = \frac{\tilde{b} \prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{M}(1-d_k z^{-1})}$
  - The roots of the numerator polynomial, c_k, are termed the ***zeros*** of *X(z)*.
  - The roots of the numerator polynomial, d_k, are termed the ***poles*** of *X(z)*.
- Example:

Find the z-transform of the signal $x[k] = -\alpha^k u[-k-1]$. Depict the ROC and locations of poles, zeros of *X(z)* in the z-plane.

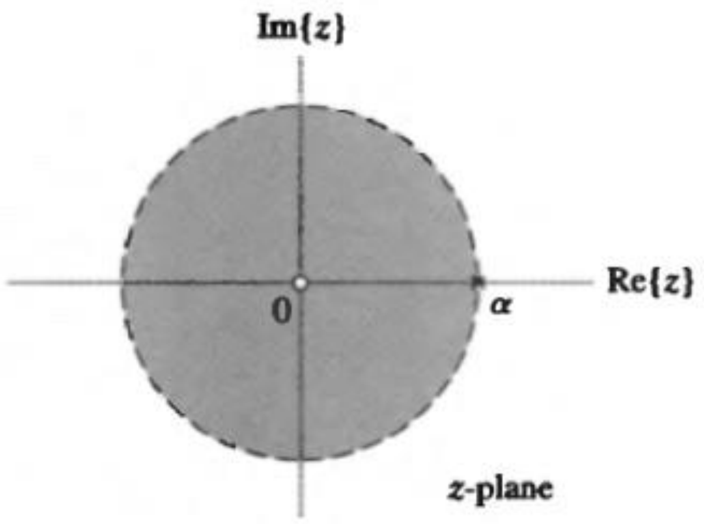
SOLUTION:

$$X(z) = \sum_{k=-\infty}^{\infty} -\alpha^k u[-k-1] z^{-k} = -\sum_{k=-\infty}^{-1} \left(\frac{\alpha}{z}\right)^k$$

$$= -\sum_{n=1}^{\infty} \left(\frac{z}{\alpha}\right)^n = 1 - \sum_{n=0}^{\infty} \left(\frac{z}{\alpha}\right)^n$$

For $|z| > |\alpha|$, the sum does not converge. For $|z| < |\alpha|$, the sum converges to $X(z) = 1 - \frac{\alpha}{\alpha - z} = \frac{z}{z-\alpha}$. The ROC of $X(z)$ is the shaded region inside the circle of radius $|\alpha|$, centered at the origin in the z-plane.

The signal has one zero at $z = 0$ and one pole at $z = \alpha$.



Properties of the ROC

- The ROC cannot contain any pole.
  - Suppose *d* is a pole of *X(z)* → *X(d)* = ±∞ → *X(z)* does not converge at *d* → *d* cannot lie in the ROC.
- The ROC for a finite-duration signal includes the entire z-plane, except possibly *z* = 0 or |*z*| = ∞ (or both).
- Suppose that $x[n]$ satisfies $|x[n]| \leq A_{-}(r_{-})^{n}, n < 0; |x(t)| \leq A_{+}(r_{+})^{n}, n \geq 0$ (i.e. $x[n]$ grows no faster than$(r_{+})^{n}$ and $(r_{-})^{n}$ for positive and negative *n*, respectively)
  - If $r_{+} < |z| < r_{-}$ then *X(z)* converges. If $r_{+} > r_{-}$, *X(z)* does not converge.
- For $x[n]$ satisfies ***exponentially bounded*** conditions above
  - If $x[n]$ is a right-sided signal (i.e. *x[n]* = 0 for *n* < 0), then the ROC of *x[n]* is of the form $|z| > r_{+}$.
  - If $x[n]$ is a left-sided signal (i.e. *x[n]* = 0 for *n* ≥ 0), then the ROC of *x[n]* is of the form $|z| < r_{-}$.
  - If $x[n]$ is a exponential two-sided signal (i.e. *x[n]* infinitely extends in both directions), then the ROC of *x[n]* is of the form $r_{+} < |z| < r_{-}$.

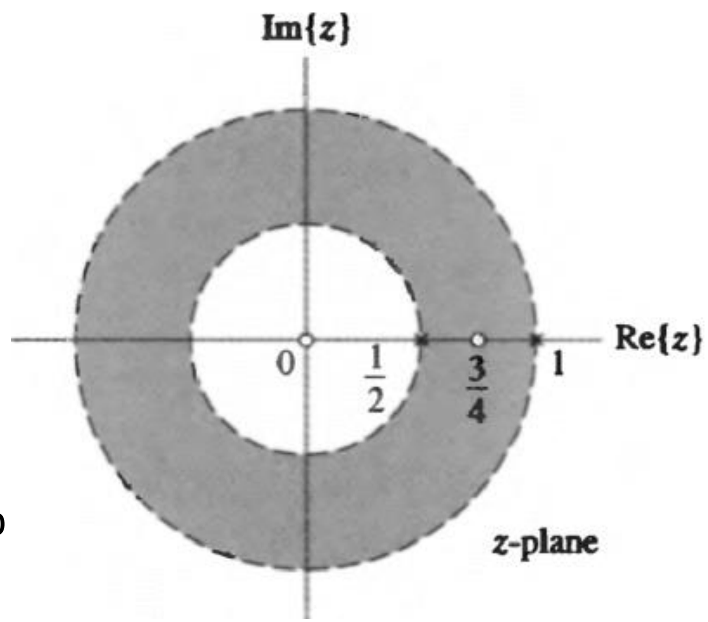
Example

- Determine the z-transform and ROC for:

  $x[n] = -u[-n-1] + \left(\frac{1}{2}\right)^n u[n].$

- **Solution**:

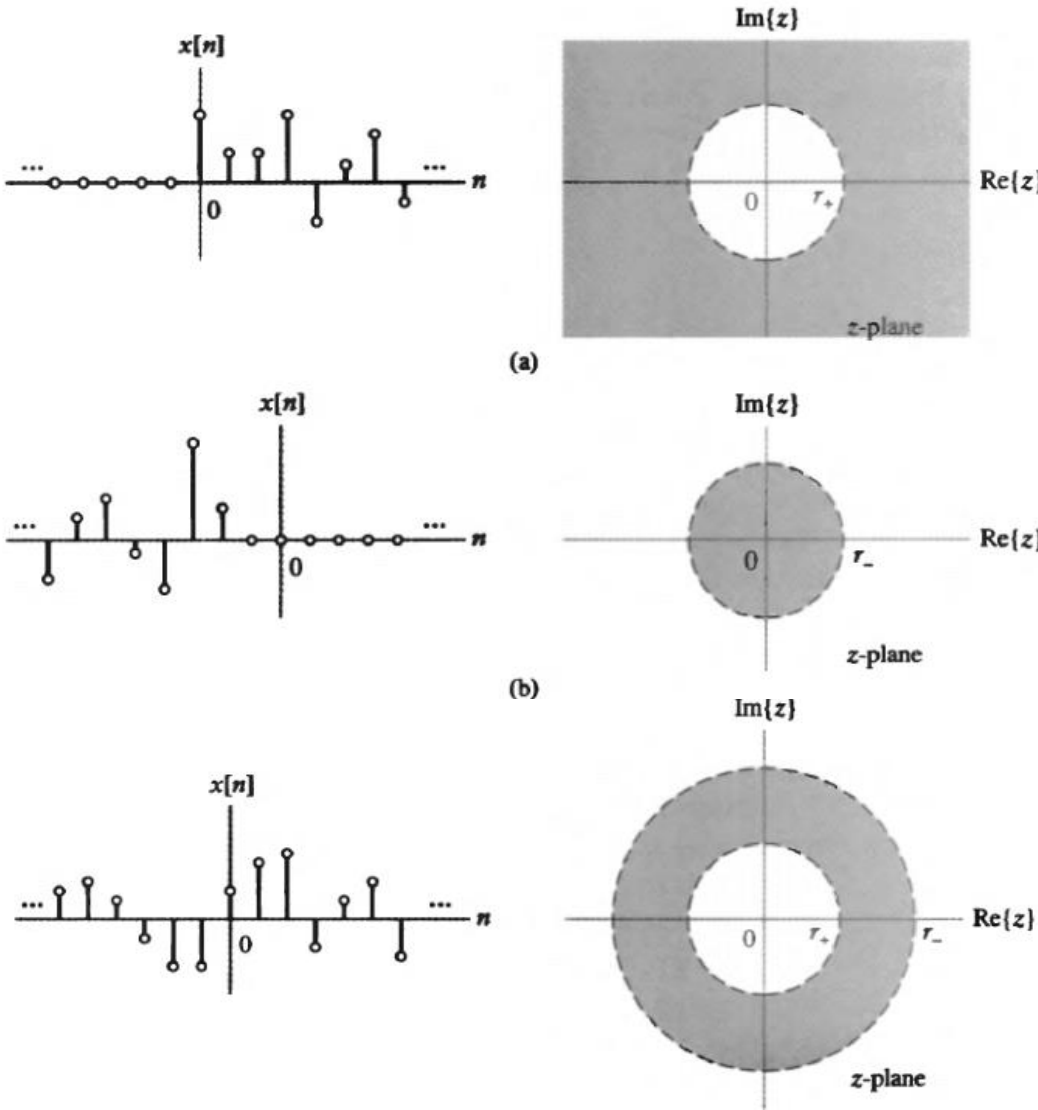
$$
\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u[n] z^{-n} - u[-n-1] z^{-n} \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} - \sum_{n=-\infty}^{-1} \left(\frac{1}{z}\right)^n \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{2z}\right)^n + 1 - \sum_{k=0}^{\infty} z^k
\end{aligned}
$$



  - Both sums must converge in order for $X(z)$ to converge $\rightarrow |z| > 1/2$ & $|z| < 1$.

  - For $1/2 < |z| < 1$ $X(z) = \frac{1}{1-\frac{1}{2z}} + 1 - \frac{1}{1-z} = \frac{z\left(2z-\frac{3}{2}\right)}{\left(z-\frac{1}{2}\right)(z-1)}$.

ROC for exponentially bounded signals



Properties of the z-transform

- **Linearity:**

$$ax[n] + by[n] \xleftrightarrow{z} aX(z) + bY(z), \quad \text{with ROC at least } R_x \cap R_y.$$

- **Time reversal:**

$$x[-n] \xleftrightarrow{z} X\left(\frac{1}{z}\right), \quad \text{with ROC } \frac{1}{R_x}.$$

- **Time shifting:**

$$x[n-n_o] \xleftrightarrow{z} z^{-n_o}X(z), \quad \text{with ROC } R_x, \text{ except possibly } z = 0 \text{ or } |z| = \infty.$$

- **Multiplication by an exponential sequence:**

$$\alpha^n x[n] \xleftrightarrow{z} X\left(\frac{z}{\alpha}\right), \quad \text{with ROC } |\alpha| R_x.$$

  - The notation $|\alpha|R_x$ means that the ROC boundaries are multiplied by $|\alpha|$: if R_x is $a < |z| < b$, then the new ROC is $|\alpha|a < |z| < |\alpha|b$.

Properties of the z-transform (cont.)

- **Convolution:**

$$\boxed{x[n] * y[n] \xleftrightarrow{z} X(z)Y(z), \quad \text{with ROC at least } R_x \cap R_y.}$$

- **Differentiation in the z-domain:**

$$\boxed{nx[n] \xleftrightarrow{z} -z\frac{d}{dz}X(z), \quad \text{with ROC } R_x.}$$

- Example: Find the z-transform of $x[n] = a^n \cos \Omega_0 n\, u[n]$ with $a \in R^+$

SOLUTION

- We have $x[n] = \frac{1}{2}e^{j\Omega_0 n}y[n] + \frac{1}{2}e^{-j\Omega_0 n}y[n]$, where $y[n] = a^n u[n] \overset{z}{\leftrightarrow} \frac{z}{z-a}$ with ROC $|z| > a$.
- Applying the property of multiplication by an exponential sequence $X(z) = \frac{1}{2}Y\left(e^{-j\Omega_0}z\right) + \frac{1}{2}Y\left(e^{j\Omega_0}z\right) = \frac{1-a\cos\Omega_0 z^{-1}}{1-2a\cos\Omega_0 z^{-1}+a^2z^{-2}}$, ROC $|z| > a$.

Some common z-transform pairs

Signal	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\|z\| < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $\alpha^n u[n]$	$\frac{1}{1-\alpha z^{-1}}$	$\|z\| > \|\alpha\|$
6. $-\alpha^n u[-n-1]$	$\frac{1}{1-\alpha z^{-1}}$	$\|z\| < \|\alpha\|$

Some common z-transform pairs (cont.)

Signal	Transform	ROC
7. $n\alpha^n u[n]$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\|z\| > \|\alpha\|$
8. $-n\alpha^n u[-n-1]$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\|z\| < \|\alpha\|$
9. $[\cos\omega_0 n]u[n]$	$\frac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\|z\| > 1$
10. $[\sin\omega_0 n]u[n]$	$\frac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\|z\| > 1$
11. $[r^n \cos\omega_0 n]u[n]$	$\frac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\|z\| > r$
12. $[r^n \sin\omega_0 n]u[n]$	$\frac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\|z\| > r$

Inversion of the z-transform

- Partial fraction expansion:
  - Bring $X(z)$ of the form $\frac{b_M z^{-M}+b_{M-1}z^{-(M-1)}+\cdots+b_1 z^{-1}+b_0}{a_0\prod_{k=1}^{N}1-d_k z^{-1}}$ to $X(s)=\sum_{k=1}^{N}\frac{A_k}{1-d_k z^{-1}}$ if all the poles d_k are distinct.
  - If a pole d_i is repeated r times, then there are r terms in the partial fraction expansion associated with that pole: $\frac{A_{i1}}{1-d_k z^{-1}},\frac{A_{i2}}{(1-d_k z^{-1})^2},\cdots,\frac{A_{ir}}{(1-d_k z^{-1})^r}$.
  - Depending on the ROC, the inverse *z*-transform associated with each term is then determined by using the appropriate transform pair:

$$A_k(d_k)^n u[n] \overset{z}{\leftrightarrow} \frac{A_k}{1-d_k z^{-1}} \text{ with ROC } |z|>d_k\text{; or}$$

$$-A_k(d_k)^n u[-n-1] \overset{z}{\leftrightarrow} \frac{A_k}{1-d_k z^{-1}} \text{ with ROC } |z|<d_k\text{; or}$$

$$A\frac{(n+1)\cdots(n+m-1)}{(m-1)!}(d_k)^n u[n] \overset{z}{\leftrightarrow} \frac{A_k}{(1-d_k z^{-1})^m} \text{ with ROC } |z|>d_k\text{; or}$$

$$-A\frac{(n+1)\cdots(n+m-1)}{(m-1)!}(d_k)^n u[-n-1] \overset{z}{\leftrightarrow} \frac{A_k}{(1-d_k z^{-1})^m} \text{ with ROC } |z|<d_k.$$

  - The linearity property indicates that the ROC of *X(z)* is the intersection of the ROCs associated with the individual terms in the partial fraction expansion → we must infer the ROC of each term from the ROC of *X(z)* to obtain the correct inverse transform.

Example

Example 7.9 Inversion by Partial-Fraction Expansion Find the inverse z-transform of

$$X(z) = \frac{1 - z^{-1} + z^{-2}}{\left(1 - \frac{1}{2}z^{-1}\right)(1 - 2z^{-1})(1 - z^{-1})}, \quad \text{with ROC } 1 < |z| < 2.$$

Solution: We use a partial-fraction expansion to write

$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{2}{1 - 2z^{-1}} - \frac{2}{1 - z^{-1}}.$$

Now we find the inverse z-transform of each term, using the relationship between the locations of the poles and the ROC of $X(z)$, each of which is depicted in Fig. 7.12. The figure shows that the ROC has a radius greater than the pole at $z = \frac{1}{2}$, so this term has the right-sided inverse transform

$$\left(\frac{1}{2}\right)^n u[n] \xleftrightarrow{z} \frac{1}{1 - \frac{1}{2}z^{-1}}.$$

The ROC also has a radius less than the pole at $z = 2$, so this term has the left-sided inverse transform

$$-2(2)^n u[-n - 1] \xleftrightarrow{z} \frac{2}{1 - 2z^{-1}}.$$

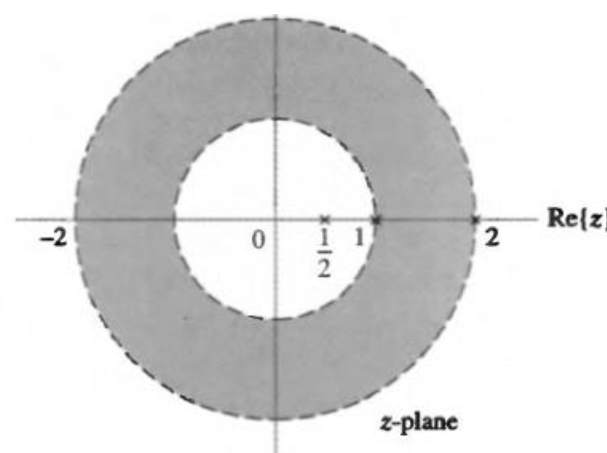


Figure 7.12 Locations of poles and ROC for Example 7.9.

Example (cont.)

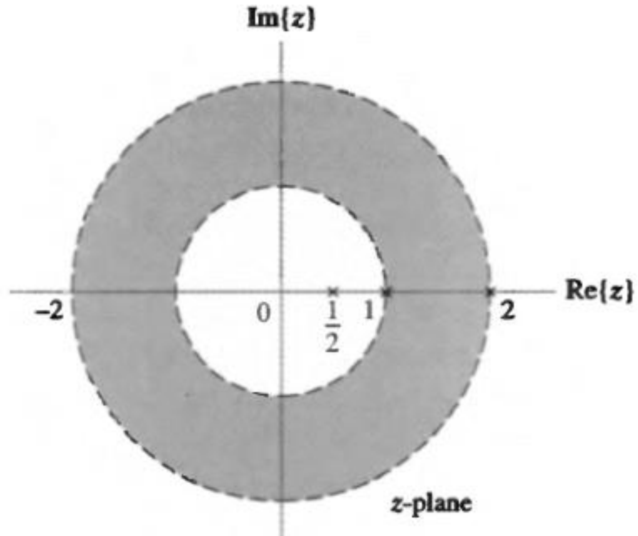


FIGURE 7.12 Locations of poles and ROC for Example 7.9.

Finally, the ROC has a radius greater than the pole at $z = 1$, so this term has the right-sided inverse z-transform

$$-2u[n] \xleftrightarrow{z} -\frac{2}{1 - z^{-1}}.$$

Combining the individual terms gives

$$x[n] = \left(\frac{1}{2}\right)^n u[n] - 2(2)^n u[-n-1] - 2u[n].$$

■

Inversion of the z-transform (cont.)

- Power series expansion:
  - Bring $X(z)$ to the form of a power series in z^{-1} or z, then the values of $x[n]$ are the coefficients associated with z^{-n}.
  - This inversion method is limited to one-sided signals only, i.e. signals with ROCs of the form $|z| > a$ or $|z| < a$. If the ROC is $|z| > a \rightarrow$ express $X(z)$ as a power series in $z^{-1} \rightarrow$ right-sided $x[n]$, and vice versa.
- **Example**: Find the inverse z-transform of $X(z) = e^{z^2}$, with ROC all z except $|z| = \infty$.
- **Solution**:
  - Using the power series representation for e^a, viz. $e^a = \sum_{k=0}^{\infty} \frac{a^k}{k!}$, we have

$$X(z) = \sum_{k=0}^{\infty} \frac{(z^2)^k}{k!} = \sum_{k=0}^{\infty} \frac{z^{2k}}{k!}$$

  - On the other hand, as $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$ by definition, we conclude that

$$x[n] = \begin{cases} 0, & n < 0 \text{ or } n \text{ odd} \\ \dfrac{1}{\left(\frac{-n}{2}\right)!}, & \text{otherwise} \end{cases}$$

The transfer function

- Derived in a similar manner to that of CT LTI systems.
  - $H(z) = \frac{Y(z)}{X(z)}$ and is called the ***transfer function*** of the DT LTI system.
  - $h[n] = \mathcal{Z}^{-1}[H(z)] = \mathcal{Z}^{-1}\left[\frac{Y(z)}{X(z)}\right]$, $y[n] = \mathcal{Z}^{-1}[H(z)X(z)]$
  - In order to uniquely determine $h[n]$, we must know the ROC. If the ROC is not known, other system characteristics such as stability or causality must be known.
- The transfer function can be obtained directly from the difference equation that describes the system.
  - Assume that the system is described by $\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$. Taking z-transform of both sides yields $\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$.
  - Rational transfer function: $H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$
  - The poles and zeros of a rational transfer function are found by factoring the numerator and denominator $H(z) = \frac{\tilde{b}\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{M}(1-d_k z^{-1})}$.

Causality and stability

- Causality: a DT LTI system is causal if $h[n] = 0\ \forall n < 0 \rightarrow$ the impulse response of a causal DT LTI system is determined from its transfer function by using right-sided inverse transform.
- Stability: a DT LTI system is causal if its impulse response is summable $\rightarrow$ the DTFT of the impulse response exists $\rightarrow$ the ROC must includes the unit circle in the z-plane.
- If a system is causal:
  - $\frac{A_k}{1-d_k z^{-1}} \overset{z}{\leftrightarrow} A_k (d_k)^n u[n]$ $\rightarrow$ the system is stable if ***all*** the poles are ***inside*** the unit circle in the *z*-plane. If there's at least one pole outside the unit circle $\rightarrow$ unstable.
- If the system is anti-causal:
  - $h[n]$ is the left-sided inverse z-transform of *H(z)*.
  - $\frac{A_k}{1-d_k z^{-1}} \overset{z}{\leftrightarrow} -A_k\, (d_k)^n u[-n-1]$ $\rightarrow$ the system is stable if ***all*** the poles are ***outside*** the unit circle in the z-plane. If there's at least one pole inside the unit circle $\rightarrow$ unstable.

Pole locations and impulse response for causal systems

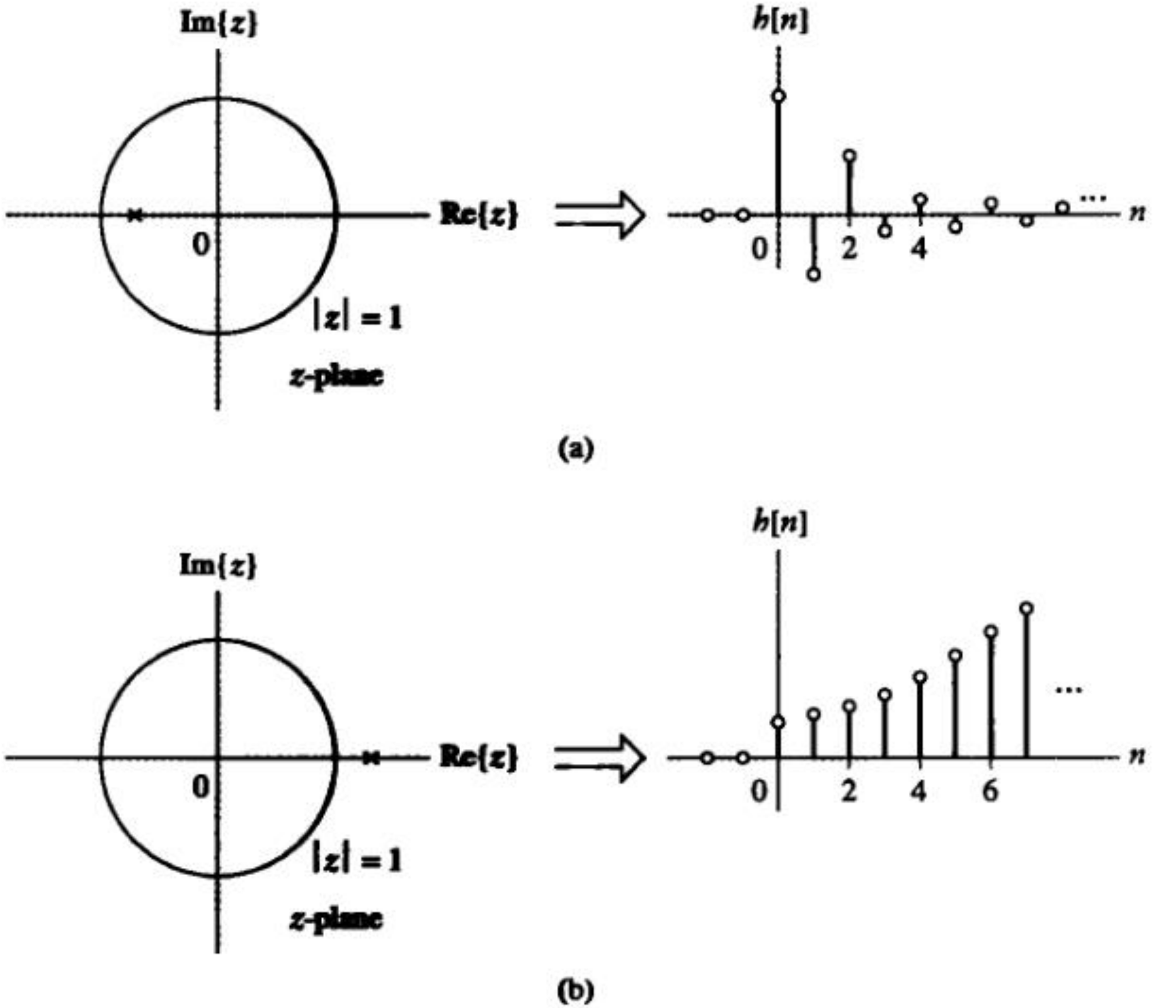


FIGURE 7.14 The relationship between the location of a pole and the impulse response characteristics for a causal system. (a) A pole inside the unit circle contributes an exponentially decaying term to the impulse response. (b) A pole outside the unit circle contributes an exponentially increasing term to the impulse response.

Pole locations and impulse response for stable systems

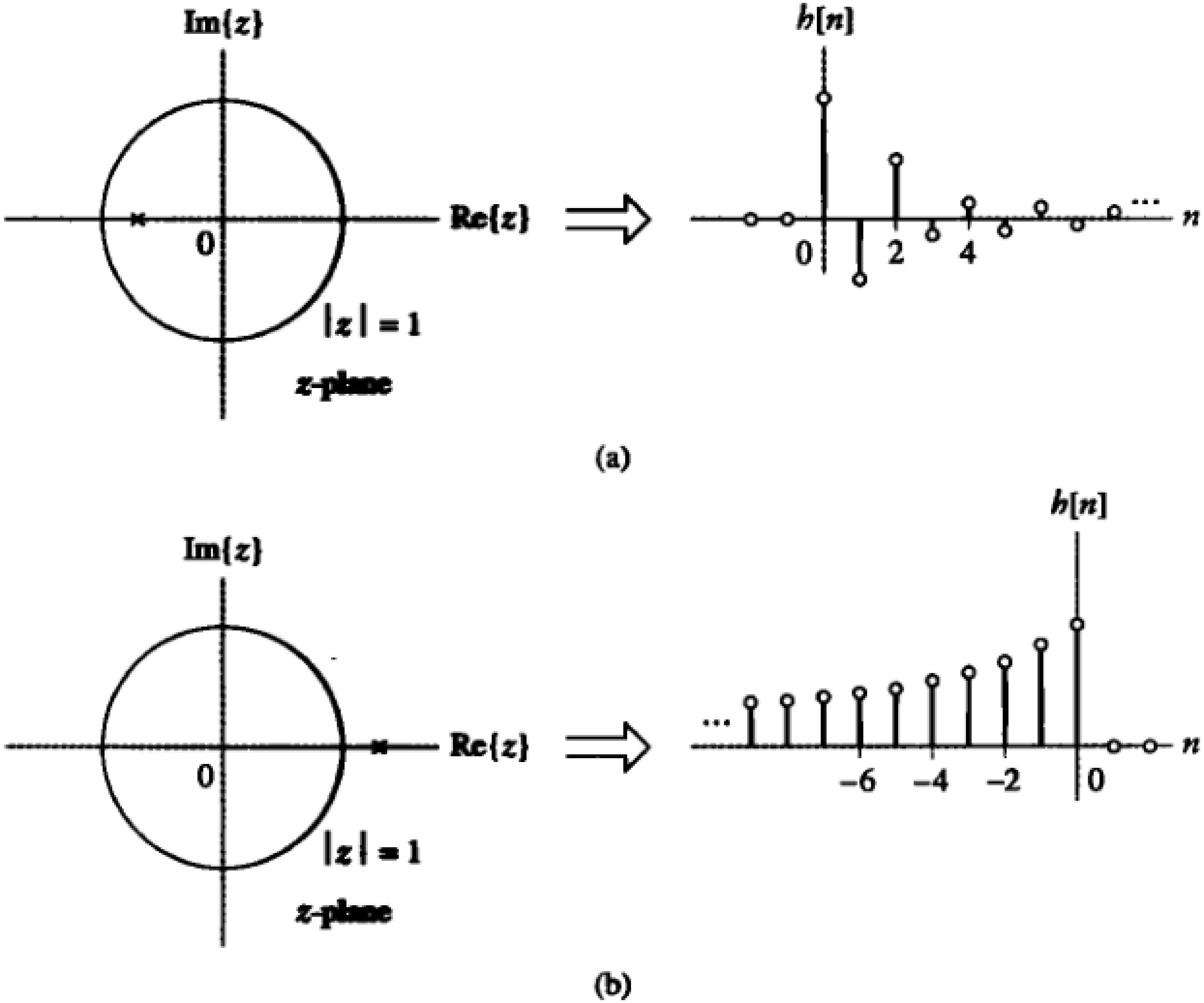
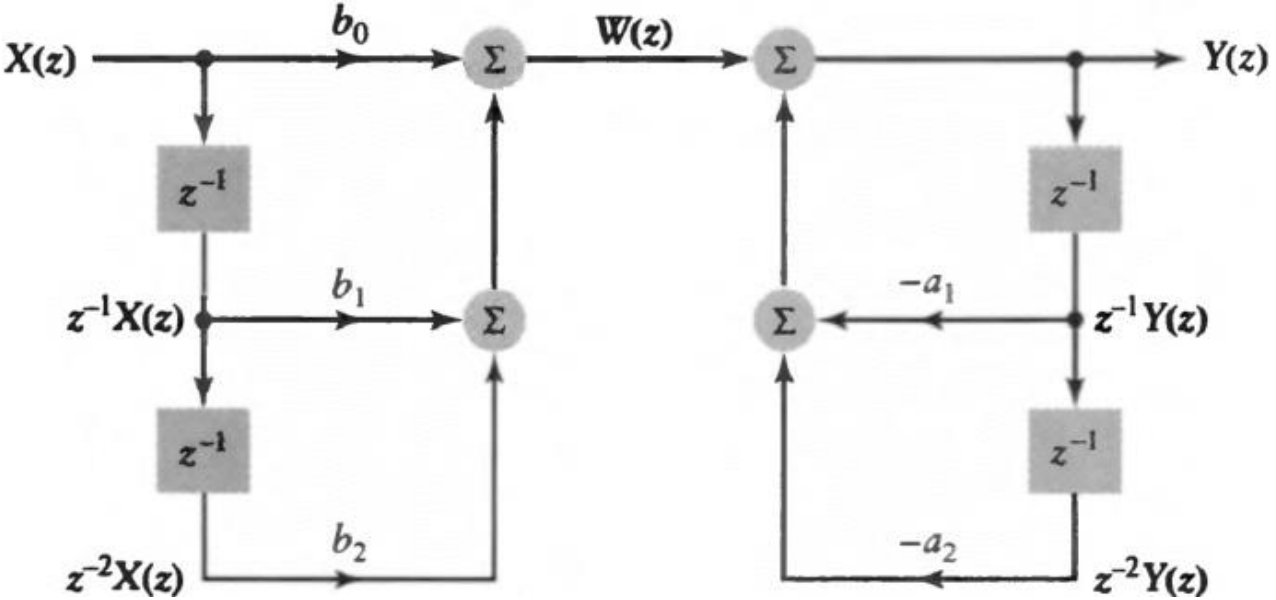


FIGURE 7.15 The relationship between the location of a pole and the impulse response characteristics for a stable system. (a) A pole inside the unit circle contributes a right-sided term to the impulse response. (b) A pole outside the unit circle contributes a left-sided term to the impulse response.

Block diagram representations of DT-LTI systems

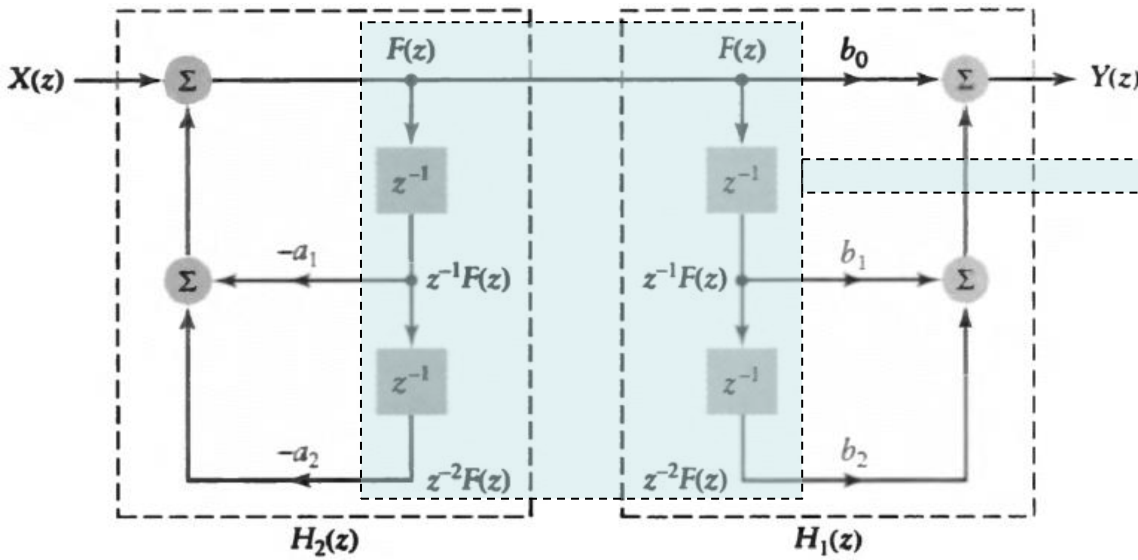
- A block diagram describes how the system's internal ***elementary operations*** are ordered.
  - The block diagram description of a system is **NOT** unique.
  - **Example**: $y[n] + a_1 y[n-1] + a_2 y[n-2] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$
  - **Solution**: $(1 + a_1 z^{-1} + a_2 z^{-2}) Y(z) = (b_0 + b_1 z^{-1} + b_2 z^{-2}) X(z)$



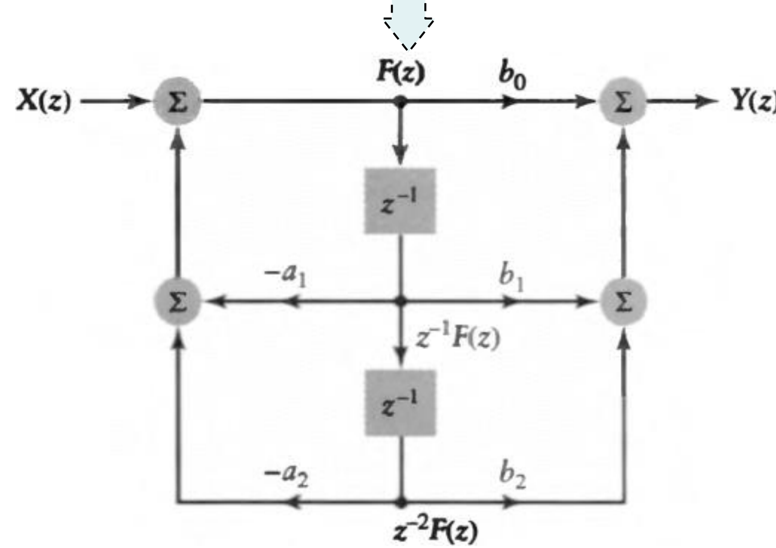
✓ z^{-1} represents the shift operator.

The direct form II

- Derived by writing the difference equation as two coupled difference equations involving an intermediate signal $f[n]$.
  - Let $H_1(z) = b_0 + b_1 z^{-1} + b_2 z^{-2}$, $H_2(z) = \frac{1}{1+a_1 z^{-1}+a_2 z^{-2}}$, and $F(z) = H_2(z)X(z)$, it's obvious that $Y(z) = H_1(z)F(z)$.



  - The z^{-1} blocks in $H_1(z)$ and $H_2(z)$ generate identical quantities → combined to get ☞

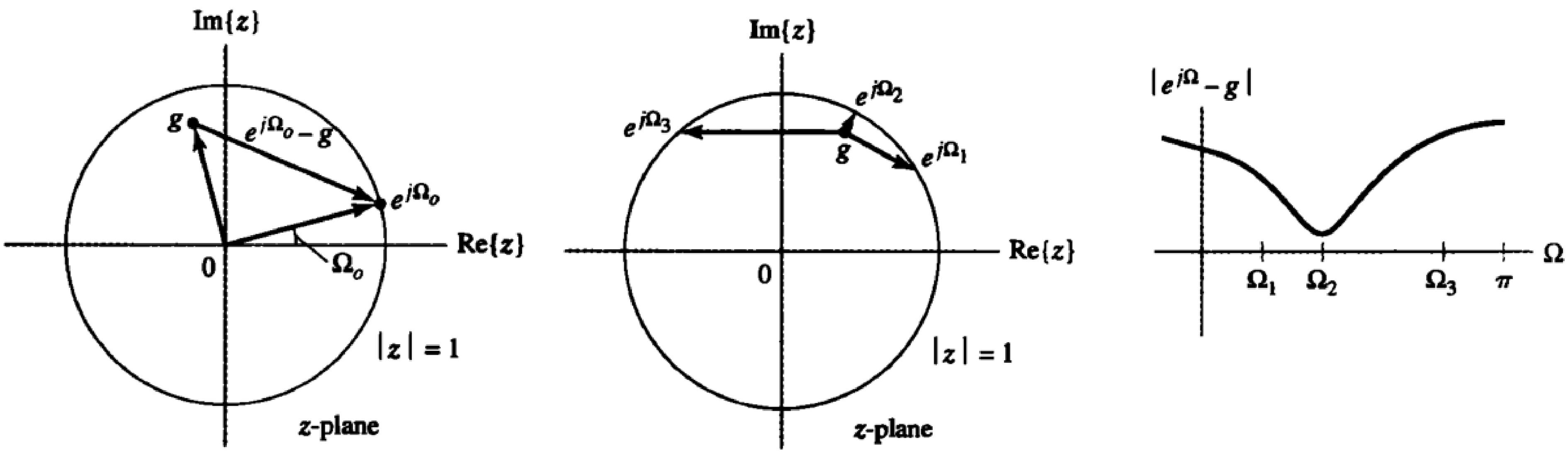


Determining the frequency response from poles and zeros

- Let $H(z) = \frac{\tilde{b} z^{-p} \prod_{k=1}^{M-p}(1-c_k z^{-1})}{z^{-l} \prod_{k=1}^{N-l}(1-d_k z^{-1})} \rightarrow H(e^{j\Omega}) = \frac{\tilde{b} e^{-jp\Omega} \prod_{k=1}^{M-p}(1-c_k e^{-j\Omega})}{e^{-jl\Omega} \prod_{k=1}^{N-l}(1-d_k e^{-j\Omega})}$.
  - The magnitude of $H(e^{j\Omega})$ at some fixed value of Ω, say, Ω_0, is defined by:

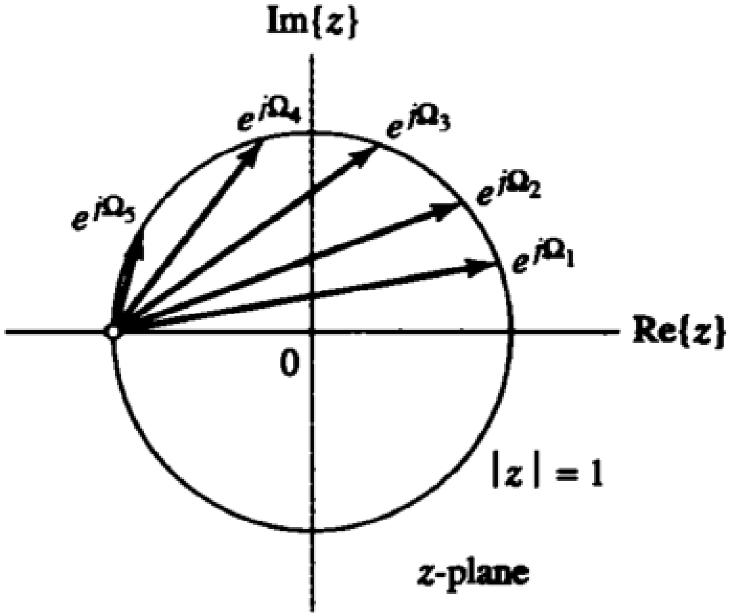
$$|H(e^{j\Omega_0})| = \frac{|\tilde{b}| \prod_{k=1}^{M-p} |e^{j\Omega_0} - c_k|}{\prod_{k=1}^{N-l} |e^{j\Omega_0} - d_k|}$$

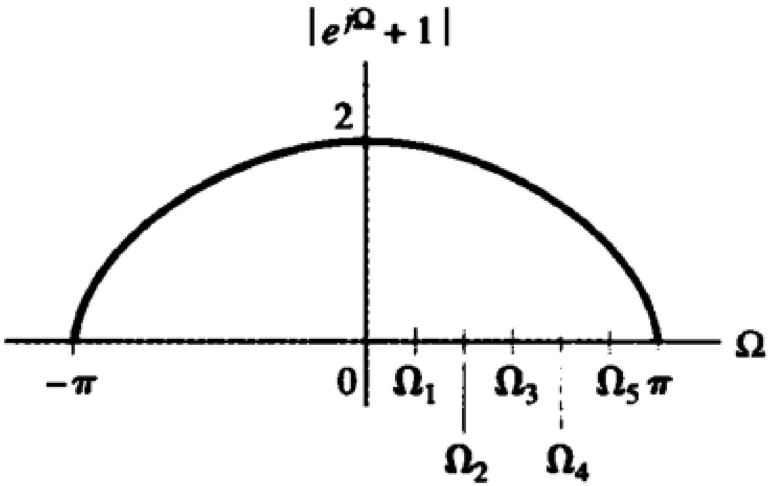
  - In the z-plane, each of the complex number $e^{j\Omega_0}$, c_k, d_k is represented by a vector from the origin to the to the corresponding point $\rightarrow e^{j\Omega_0} - g$ is a vector from the point g to the point $e^{j\Omega_0}$ (g can be either a pole or zero).
  - The frequency response is evaluated by the contribution of all vectors $e^{j\Omega_0} - g$.



Example

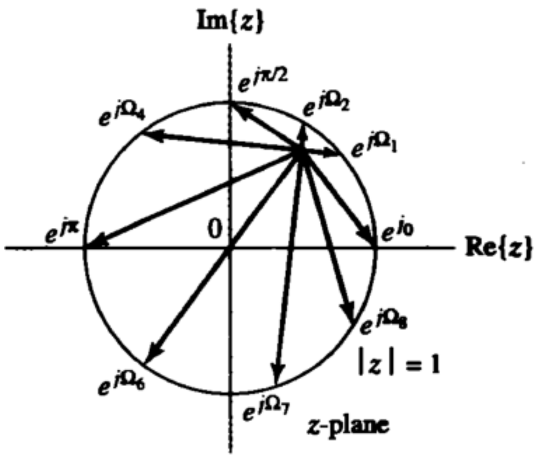
- Sketch the magnitude response for $H(z) = \frac{1+z^{-1}}{\left(1-0.9e^{j\frac{\pi}{4}}z^{-1}\right)\left(1-0.9e^{-j\frac{\pi}{4}}z^{-1}\right)}$.
- **Solution**:
  - One zero at $z = -1$ and two poles at $z = 0.9e^{j\frac{\pi}{4}}$ and $z = 0.9e^{-j\frac{\pi}{4}}$.
  - The contribution of the zero to the magnitude response can be evaluated as follows

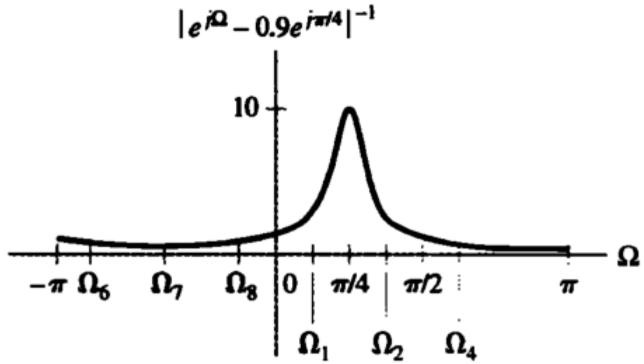


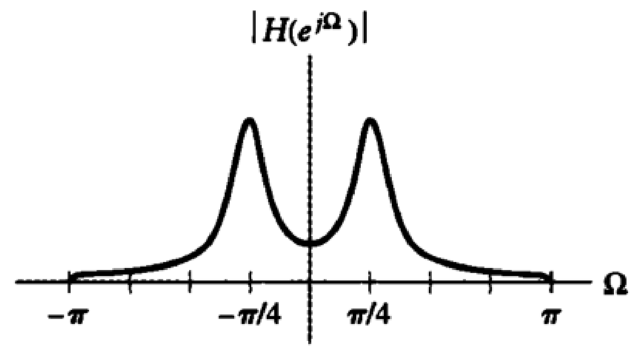


Example solution

Contribution of the pole at $z = 0.9e^{j\frac{\pi}{4}}$

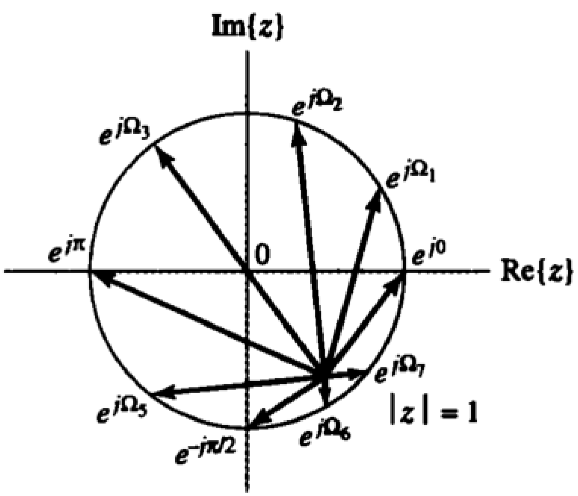


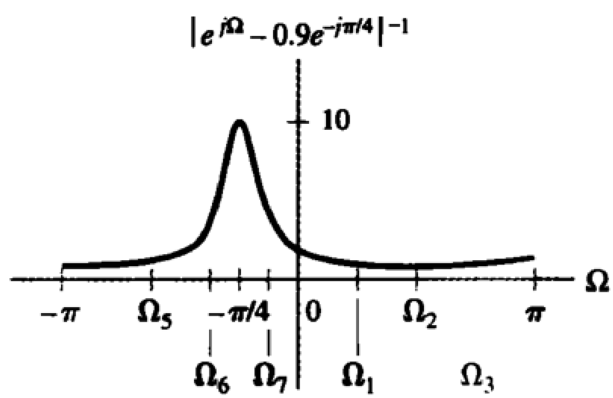




The overall magnitude response

Contribution of the pole at $z = 0.9e^{-j\frac{\pi}{4}}$





Unilateral z-transform

- There are many applications where the signals involved are causal
  - It is advantageous to define the unilateral z-transform that works only on the non-negative time portion of the signal → no need to consider the ROC.
- Definition

$$F(z) = \sum_{k=0}^{\infty} f[k] z^{-k} \qquad (3)$$

  - The inverse transform remains the same as in the bilateral case.
- Properties: similar to those of the bilateral transform → self-study.
  - One important exception is the ***time shift property***. The unilateral version for a delayed signal is

$$\boxed{x[n-k] \overset{z_u}{\longleftrightarrow} x[-k] + x[-k+1]z^{-1} + \cdots + x[-1]z^{-k+1} + z^{-k}X(z) \text{ for } k > 0}$$

  - and for an advanced signal is

$$\boxed{x[n+k] \overset{z_u}{\longleftrightarrow} -x[0]z^{k} - x[1]z^{k-1} - \cdots - x[k-1]z + z^{k}X(z) \text{ for } k > 0}$$

  - Application: solving difference equations with initial conditions.

Solving difference equations with initial conditions via unilateral z-transform

- Taking unilateral z-transform of both sides of a difference equation
  - Use algebra to obtain the z-transform of the solution, and then find the inverse z-transform.
- Consider $\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$
  - Taking unilateral z-transform:

$$Y(z)\underbrace{\sum_{k=0}^{N} a_k z^{-k}}_{A(z)} + \underbrace{\sum_{m=0}^{N-1} \sum_{k=m+1}^{N} a_k y[-k+m] z^{-m}}_{C(z)} = X(z)\underbrace{\sum_{k=0}^{M} b_k z^{-k}}_{B(z)}$$

  - Solving for z-transform of the solution: $Y(z) = \frac{B(z)}{A(z)} - \frac{C(z)}{A(z)}$
  - Find $\mathcal{Z}^{-1}[Y(z)]$ by partial fraction expansion or power series expansion.
- **Example**: find the forced and natural responses of the system

$$y[n] + 3y[n-1] = x[n] + x[n-1]$$

if the input is $x[n] = \left(\frac{1}{2}\right)^n u[n]$ and the initial condition is $y[-1] = 2$

Example solution

- Taking unilateral *z*-transform of both sides of a difference equation

$$Y(z)\underbrace{\left(1+3z^{-1}\right)}_{A(z)}+\underbrace{3y[-1]}_{C(z)}=X(z)\underbrace{\left(1+z^{-1}\right)}_{B(z)}$$

- Solving for z-transform of the solution:

$$Y(z)=\frac{1+z^{-1}}{1+3z^{-1}}\frac{1}{1-\left(\frac{1}{2}\right)z^{-1}}-\frac{6}{1+3z^{-1}}$$

- Taking partial fraction expansion for *Y(z)*:

$$Y(z)=\left(\frac{4/7}{1+3z^{-1}}+\frac{3/7}{1-\left(\frac{1}{2}\right)z^{-1}}\right)-\frac{6}{1+3z^{-1}}$$

- Taking inverse z-transform of *Y(z)* yields
  - Forced response $y^{(f)}[n]=\frac{4}{7}(-3)^n u[n]+\frac{3}{7}\left(\frac{1}{2}\right)^n u[n]$
  - Natural response $y^{(n)}[n]=-6(-3)^n u[n]$
  - Total response $y[n]=-\frac{38}{7}(-3)^n u[n]+\frac{3}{7}\left(\frac{1}{2}\right)^n u[n]$