ELT2035 Signals & Systems

Lesson 9: Signals and systems analysis practice

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Exercise 1

Sketch the magnitude and phase spectra of the following signals

a. $x(t) = 2e^{5t}u(t)$

b. $x(t) = -2e^{-5t}u(t)$

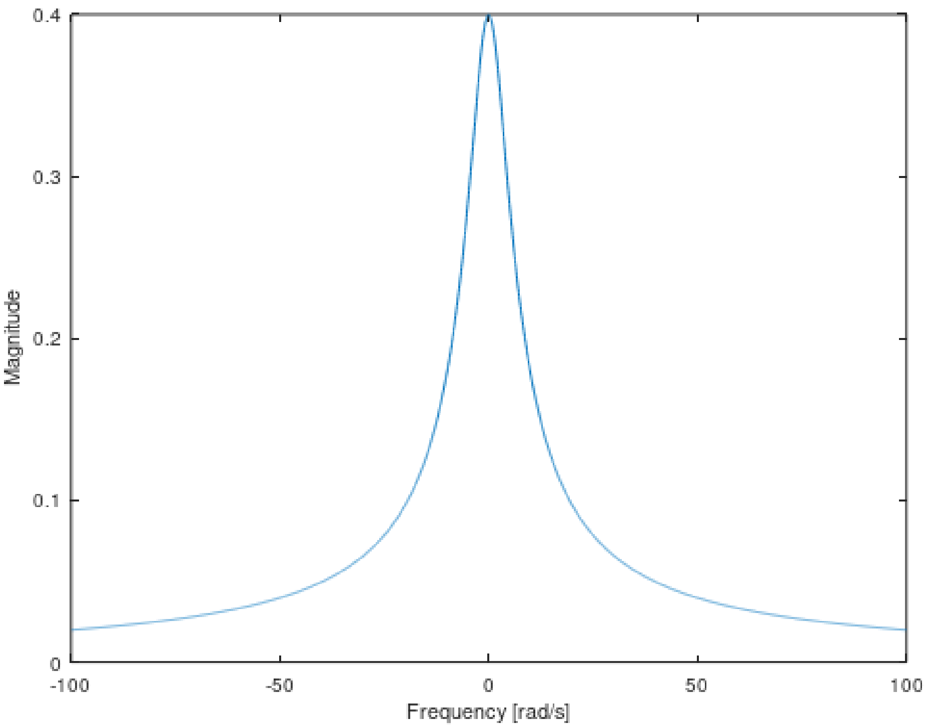
SOLUTION

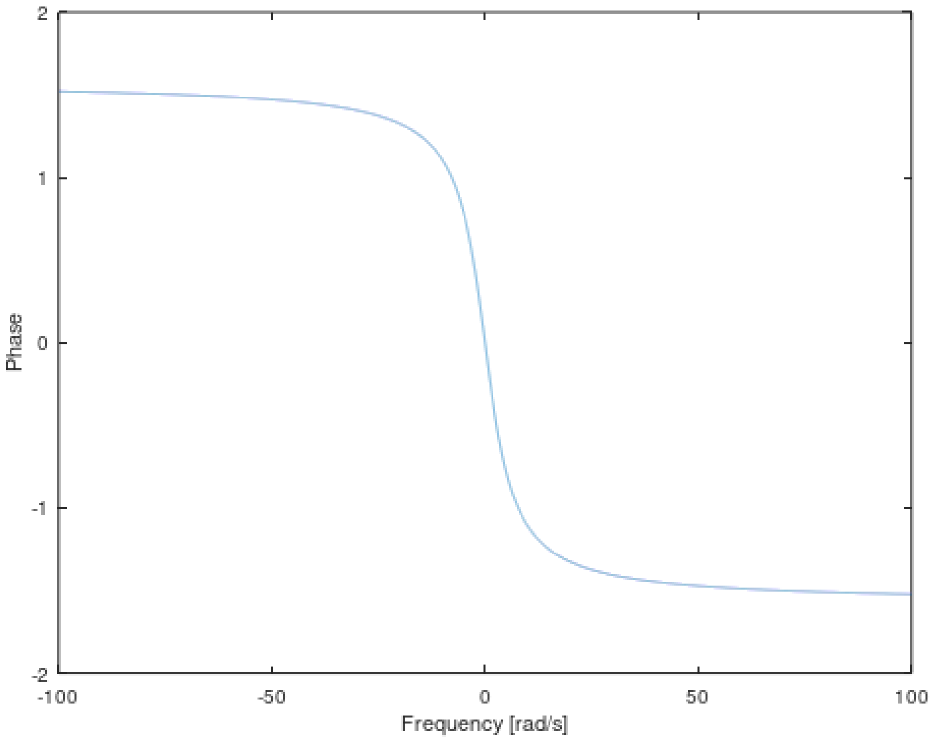
a. x(t) is unbounded → its FT is not defined (remember Dirichlet conditions in Lesson 6?)

- Dirichlet conditions for pointwise convergence:
  1. x(t) is absolutely integrable, i.e. $\int_{-\infty}^{\infty}|x(t)|dt < \infty$
  2. x(t) has a finite number of extrema and discontinuities in any finite interval
  3. The size of each discontinuity is finite

b. We have: $X(\omega) = \int_{-\infty}^{\infty} -2e^{-5t}u(t)e^{-j\omega t}dt = \int_{0}^{\infty} -2e^{(-5-j\omega)t}dt = \frac{-2}{-5-j\omega}e^{(-5-j\omega)t}\Big|_0^{\infty} = \frac{-2}{j\omega+5} = -\frac{10}{\omega^2+25} + j\frac{2\omega}{\omega^2+25}$. Hence $|X(j\omega| = \frac{2}{\sqrt{\omega^2+25}}$, $\angle X(j\omega) = -\tan^{-1}\frac{\omega}{5}$. Sketches of $|X(j\omega|$ and $\angle X(j\omega)$ are:

Exercise 1 solution





Exercise 2

Find the frequency domain representation of the following signals

a. $x[n] = a^n u[n]$, $|a| < 1$.

b. $x[n] = \left(\frac{3}{4}\right)^n u[n-4]$.

SOLUTION

a. By definition, the DTFT of $x[n]$ is: $X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = \sum_{n=0}^{\infty} a^n e^{-j\Omega n} = \sum_{n=0}^{\infty}\left(ae^{-j\Omega}\right)^n$. Since $\left|ae^{-j\Omega}\right| = a < 1$, the (complex) power series converges to $\lim_{N\to\infty} \frac{1-\left(ae^{-j\Omega}\right)^{N+1}}{1-ae^{-j\Omega}} = \frac{1}{1-ae^{-j\Omega}}$.

b. Since $x[n] = \left(\frac{3}{4}\right)^4 \left(\frac{3}{4}\right)^{n-4} u[n-4]$, applying linearity and time shift properties of the DTFT to the signal $x'[n] = \left(\frac{3}{4}\right)^n u[n]$ we arrive at

$$X(e^{j\Omega}) = \left(\frac{3}{4}\right)^4 e^{-j4\Omega} X'(e^{j\Omega}).$$

Substituting the result from part a for $X'(e^{j\Omega})$ yields $X(e^{j\Omega}) = \frac{\left(\frac{3e^{-j\Omega}}{4}\right)^4}{1-\frac{3e^{-j\Omega}}{4}}$.

Exercise 3

Find the frequency domain representation of the following signals, given that $y(t) = e^{-at}u(t) \overset{FT}{\longleftrightarrow} Y(j\omega) = \frac{1}{j\omega + a}$.

a. $x(t) = e^{-a|t|}$,

b. $x(t) = \sin(2\pi t)\, e^{-t}u(t)$

SOLUTION

a. We have $x(t) = y(-t) + y(t)$. Applying linearity and time scaling properties of the FT to $x(t)$ yields $X(j\omega) = Y(-j\omega) + Y(j\omega) = \frac{1}{a - j\omega} + \frac{1}{a + j\omega} = \frac{2a}{a^2 + \omega^2}$.

b. We have:

$$x(t) = \frac{1}{2j}\left(e^{j2\pi t} - e^{-j2\pi t}\right)e^{-t}u(t) = \frac{1}{2j}e^{j2\pi t}e^{-t}u(t) - \frac{1}{2j}e^{j2\pi t}e^{-t}u(t).$$

Applying linearity and frequency shift properties of the FT to $x(t)$ yields

$$X(j\omega) = \frac{1}{2j}Y(j(\omega - 2\pi)) + \frac{1}{2j}Y(j(\omega + 2\pi)) = \frac{1}{2j}\left[\frac{1}{1 + j(\omega - 2\pi)} - \frac{1}{1 + j(\omega + 2\pi)}\right].$$

Exercise 4

a. Find the inverse FT of $X(j\omega) = \frac{j\omega}{(1+j\omega)^2}$,

b. Find the FT of $x(t) = \frac{1}{(2+jt)^2}$

SOLUTION

a. Since $e^{-t}u(t) \overset{FT}{\longleftrightarrow} \frac{1}{j\omega+1}$, applying the differentiation in frequency domain results in $te^{-t}u(t) \overset{FT}{\longleftrightarrow} \frac{1}{(j\omega+1)^2}$. Applying differentiation in time domain yields $x(t) = \frac{d}{dt}[te^{-at}u(t)] = (1-t)e^{-t}u(t)$.

b. Since $te^{-2t}u(t) \overset{FT}{\longleftrightarrow} \frac{1}{(j\omega+2)^2}$, applying duality property of the FT yields

- If $f(t) \overset{FT}{\longleftrightarrow} F(j\omega)$ then $F(jt) \overset{FT}{\longleftrightarrow} 2\pi f(-\omega)$.
  - Proof: HW

$$X(j\omega) = 2\pi(-\omega)e^{-2(-\omega)}u(-\omega) = -2\pi\omega e^{2\omega}u(-\omega).$$

Exercise 5

Find the inverse FT of the signal $X(j\omega) = \frac{\pi\delta(\omega)}{j\omega+2}$.

SOLUTION

- Since $1 \overset{FT}{\longleftrightarrow} 2\pi\delta(\omega)$, $e^{-2t}u(t) \overset{FT}{\longleftrightarrow} \frac{1}{j\omega+2}$, applying linearity and convolution properties of the FT yields

$$x(t) = \frac{1}{2} * e^{-2t}u(t) = \int_{-\infty}^{\infty} \frac{1}{2} e^{-2\tau}u(\tau)d\tau = \frac{1}{2}\int_{0}^{\infty} e^{-2\tau}u(\tau)d\tau = \frac{1}{4}.$$

- **System interpretation:**

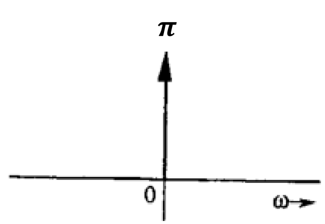
Time domain: *x(t)*

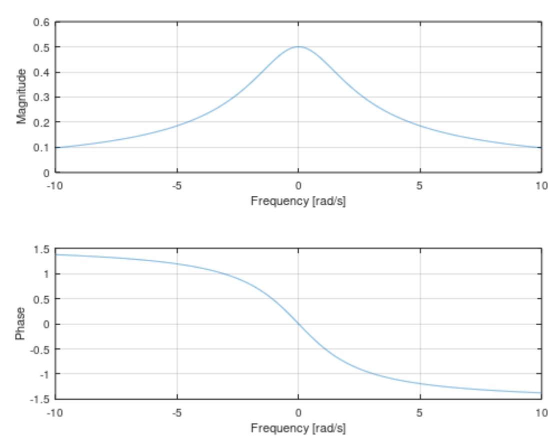
Frequency domain *X(jω)*

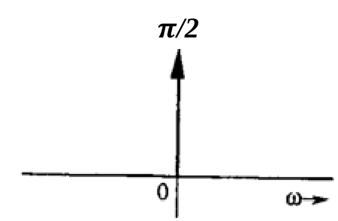
System H[.]

Time domain: y*(t)*

Frequency domain *Y(jω)*







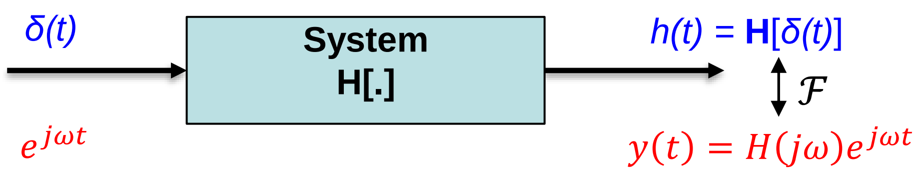
Exercise 6

A system is described by input-output relationship $y(t) = x(t+1) + 2x(t) + x(t-2)$. Find:

a. The impulse response of the system

b. The frequency response of the system

SOLUTION



a. Let $x(t) = \delta(t)$, the impulse response is

$$h(t) = \delta(t+1) + 2\delta(t) + \delta(t-1).$$

b. Let $x(t) = e^{j\omega t}$, the output is $y(t) = e^{j\omega(t+1)} + 2e^{j\omega t} + e^{j\omega(t-2)} = \left(e^{j\omega} + 2 + e^{-j2\omega}\right)e^{j\omega t} \rightarrow$ the frequency response is $y(t) = H(j\omega)e^{j\omega t} = e^{j\omega} + 2 + e^{-j2\omega}$.

Alternatively: $H(j\omega) = \mathcal{F}[h(t)] = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty}[\delta(t+1) + 2\delta(t) + \delta(t-1)]e^{-j\omega t}dt = \int_{-\infty}^{\infty} e^{-j\omega(-1)}\delta(t+1)dt + \int_{-\infty}^{\infty} e^{-j\omega 0}\delta(t)dt + \int_{-\infty}^{\infty} e^{-j\omega 2}\delta(t-2)dt = e^{j\omega} + 2 + e^{-j2\omega}$.

Exercise 7

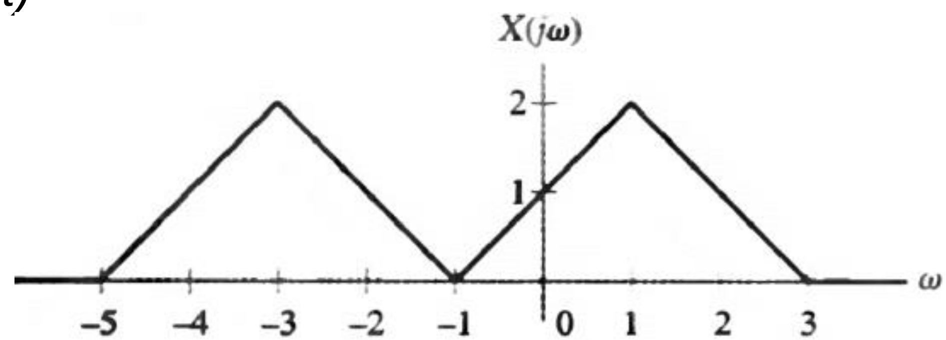
A signal's spectrum is presented in the below figure. Evaluate the following quantities without computing *x(t)*

a. $x(0)$

b. $\int_{-\infty}^{\infty} x(t)dt$

c. $\int_{-\infty}^{\infty} |x(t)|^2 dt$

d. $\int_{-\infty}^{\infty} x(t)e^{j3t}dt$



SOLUTION

a. We have $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \Longrightarrow x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)d\omega = \frac{4}{\pi}$.

b. We have $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \Longrightarrow \int_{-\infty}^{\infty} x(t)dt = X(j0) = 1$.

c. According to Parseval's theorem $\int_{-\infty}^{\infty}|x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|X(j\omega)|^2 d\omega =$

$$\frac{1}{2\pi}\left[\int_{-5}^{-3}(\omega+5)^2 d\omega + \int_{-3}^{-1}(-\omega+1)^2 d\omega + \int_{-1}^{1}(\omega+1)^2 d\omega + \int_{1}^{3}(-\omega+3)^2 d\omega\right]$$

$$= \frac{16}{3\pi}.$$

d. We have $\int_{-\infty}^{\infty} x(t)e^{j3t}dt = X(j(-3)) = 2$.