Data Structures and Algorithms

Priority Queues

Outline

- Priority Queues
- Heaps
- Adaptable Priority Queues

Priority Queues

Priority Queue ADT

- A priority queue stores a collection of entries
- Each **entry** is a pair (key, value)
- Main methods of the Priority Queue ADT
  - insert(k, x)
    inserts an entry with key k and value x
  - removeMin()
    removes and returns the entry with smallest key
- Additional methods
  - min()
    returns, but does not remove, an entry with smallest key
  - size(), isEmpty()
- Applications:
  - Standby flyers
  - Auctions
  - Stock market

Total Order Relations

- Keys in a priority queue can be arbitrary objects on which an order is defined
- Two distinct entries in a priority queue can have the same key

- Mathematical concept of total order relation $\leq$
  - Reflexive property: $x \leq x$
  - Antisymmetric property: $x \leq y \wedge y \leq x \Rightarrow x = y$
  - Transitive property: $x \leq y \wedge y \leq z \Rightarrow x \leq z$

Entry ADT

- An **entry** in a priority queue is simply a key-value pair
- Priority queues store entries to allow for efficient insertion and removal based on keys
- Methods:
  - key(): returns the key for this entry
  - value(): returns the value associated with this entry
- As a Java interface:

```java
/**
 * Interface for a key-value
 * pair entry
 **/
public interface Entry {
    public Object key();
    public Object value();
}
```

Comparator ADT

- A comparator encapsulates the action of comparing two objects according to a given total order relation
- A generic priority queue uses an auxiliary comparator
- The comparator is external to the keys being compared
- When the priority queue needs to compare two keys, it uses its comparator

- The primary method of the Comparator ADT:
  - compare(x, y): Returns an integer i such that $i < 0$ if $a < b$, $i = 0$ if $a = b$, and $i > 0$ if $a > b$; an error occurs if a and b cannot be compared.

Example Comparator

- Lexicographic comparison of 2-D points:

```
/** Comparator for 2D points under the
    standard lexicographic order. */
public class Lexicographic implements
    Comparator {
  int xa, ya, xb, yb;
  public int compare(Object a, Object b)
    throws ClassCastException {
     xa = ((Point2D) a).getX();
     ya = ((Point2D) a).getY();
     xb = ((Point2D) b).getX();
     yb = ((Point2D) b).getY();
     if (xa != xb)
            return (xb - xa);
     else
            return (yb - ya);
  }
}
```

- Point objects:

```
/** Class representing a point in the
    plane with integer coordinates */
public class Point2D {
  protected int xc, yc; // coordinates
  public Point2D(int x, int y) {
    xc = x;
    yc = y;
  }
  public int getX() {
        return xc;
  }
  public int getY() {
        return yc;
  }
}
```

Priority Queue Sorting

- We can use a priority queue to sort a set of comparable elements
  1. Insert the elements one by one with a series of insert operations
  2. Remove the elements in sorted order with a series of removeMin operations
- The running time of this sorting method depends on the priority queue implementation

Algorithm *PQ-Sort*(*S, C*)
    Input sequence *S*, comparator *C* for the elements of *S*
    Output sequence *S* sorted in increasing order according to *C*
    P ← priority queue with comparator *C*
    while ¬*S.isEmpty* ()
        e ← *S.removeFirst* ()
        P.insert (*e*, *0*)
    while ¬*P.isEmpty*()
        e ← *P.removeMin*()*.key*()
        S.insertLast(*e*)

Sequence-based Priority Queue

- Implementation with an unsorted list

- Performance:
  - insert takes $O(1)$ time since we can insert the item at the beginning or end of the sequence
  - removeMin and min take $O(n)$ time since we have to traverse the entire sequence to find the smallest key

- Implementation with a sorted list

- Performance:
  - insert takes $O(n)$ time since we have to find the place where to insert the item
  - removeMin and min take $O(1)$ time, since the smallest key is at the beginning

Selection-Sort

- Selection-sort is the variation of PQ-sort where the priority queue is implemented with an unsorted sequence
- Running time of Selection-sort:
  1. Inserting the elements into the priority queue with n insert operations takes $O(n)$ time
  2. Removing the elements in sorted order from the priority queue with n removeMin operations takes time proportional to

$$1 + 2 + \ldots + n$$

- Selection-sort runs in $O(n^2)$ time

Selection-Sort Example

	***Sequence** S*	***Priority Queue** P*
Input:	(7,4,8,2,5,3,9)	()
Phase 1		
(a)	(4,8,2,5,3,9)	(7)
(b)	(8,2,5,3,9)	(7,4)
..	.. ..	
.	. .	
(g)	()	(7,4,8,2,5,3,9)
Phase 2		
(a)	(2)	(7,4,8,5,3,9)
(b)	(2,3)	(7,4,8,5,9)
(c)	(2,3,4)	(7,8,5,9)
(d)	(2,3,4,5)	(7,8,9)
(e)	(2,3,4,5,7)	(8,9)
(f)	(2,3,4,5,7,8)	(9)
(g)	(2,3,4,5,7,8,9)	()

Insertion-Sort

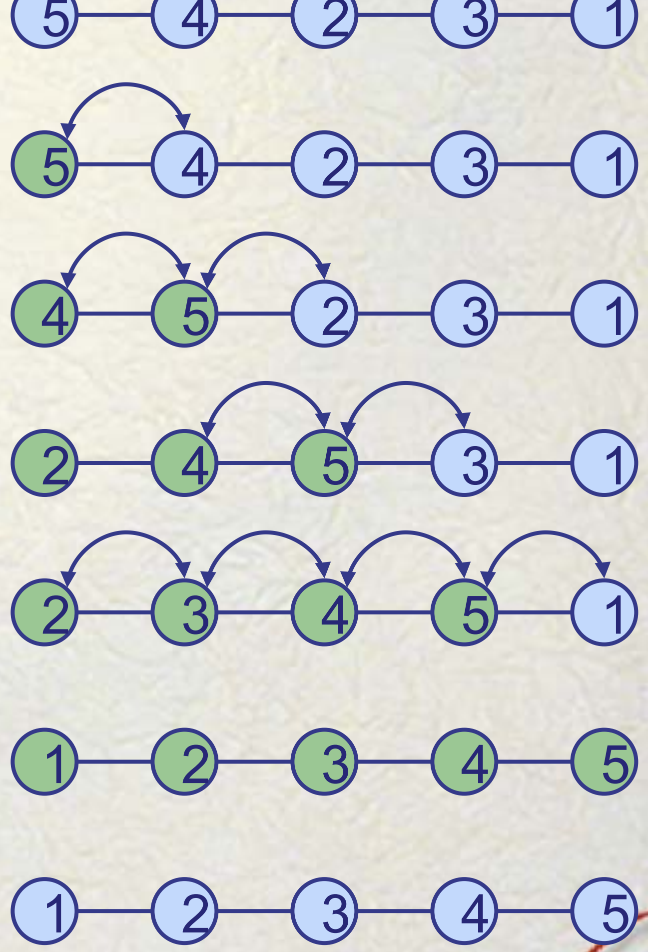
- Insertion-sort is the variation of PQ-sort where the priority queue is implemented with a sorted sequence
- Running time of Insertion-sort:
  1. Inserting the elements into the priority queue with n insert operations takes time proportional to
     $$1 + 2 + \ldots + n$$
  2. Removing the elements in sorted order from the priority queue with a series of n removeMin operations takes $O(n)$ time
- Insertion-sort runs in $O(n^2)$ time

Insertion-Sort Example

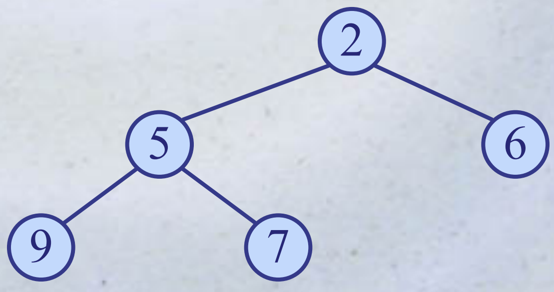
	Sequence S	*Priority queue P*
Input:	(7,4,8,2,5,3,9)	()
Phase 1		
(a)	(4,8,2,5,3,9)	(7)
(b)	(8,2,5,3,9)	(4,7)
(c)	(2,5,3,9)	(4,7,8)
(d)	(5,3,9)	(2,4,7,8)
(e)	(3,9)	(2,4,5,7,8)
(f)	(9)	(2,3,4,5,7,8)
(g)	()	(2,3,4,5,7,8,9)
Phase 2		
(a)	(2)	(3,4,5,7,8,9)
(b)	(2,3)	(4,5,7,8,9)
..	..	..
.	.	.
(g)	(2,3,4,5,7,8,9)	()

In-place Insertion-sort

- Instead of using an external data structure, we can implement selection-sort and insertion-sort in-place
- A portion of the input sequence itself serves as the priority queue
- For in-place insertion-sort
  - We keep sorted the initial portion of the sequence
  - We can use swaps instead of modifying the sequence



Heaps



Recall Priority Queue ADT

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- Main methods of the Priority Queue ADT
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- Applications:
  - Standby flyers
  - Auctions
  - Stock market

Recall Priority Queue Sorting

- We can use a priority queue to sort a set of comparable elements
  - Insert the elements with a series of insert operations
  - Remove the elements in sorted order with a series of removeMin operations
- The running time depends on the priority queue implementation:
  - Unsorted sequence gives selection-sort: $O(n^2)$ time
  - Sorted sequence gives insertion-sort: $O(n^2)$ time
- Can we do better?

Algorithm ***PQ-Sort***(***S***, ***C***)

  Input sequence ***S***, comparator ***C*** for the elements of ***S***

  Output sequence ***S*** sorted in increasing order according to ***C***

  P ← priority queue with comparator ***C***

  while ¬***S.isEmpty*** ()

    e ← ***S.remove*** (***S. first*** ())

    P.insertItem(***e***, ***e***)

  while ¬***P.isEmpty***()

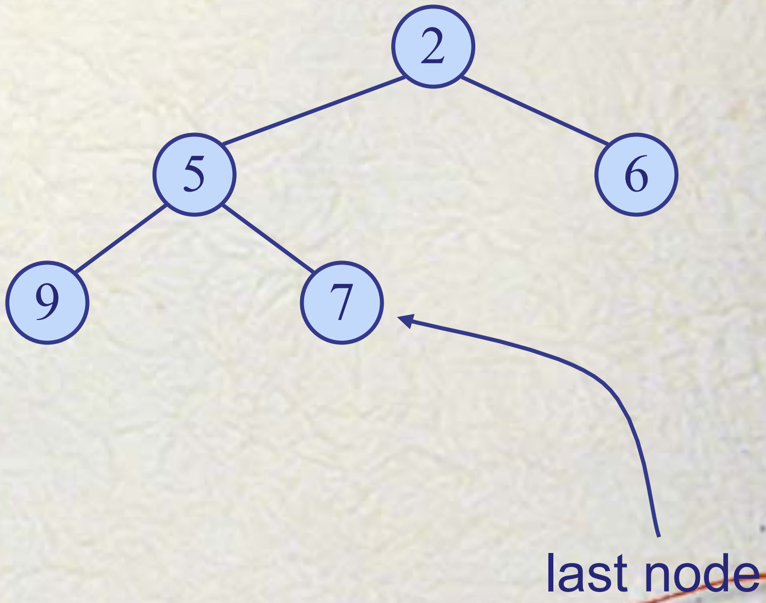
    e ← ***P.removeMin***()

    S.insertLast(***e***)

Heaps

- A heap is a binary tree storing keys at its nodes and satisfying the following properties:
  - Heap-Order: for every internal node v other than the root,
    $key(v) \geq key(parent(v))$
  - Complete Binary Tree: let h be the height of the heap
    - for $i = 0, \ldots, h - 1$, there are 2^i nodes of depth i
    - at depth h, the internal nodes are to the left of the external nodes

- The last node of a heap is the rightmost node of depth h

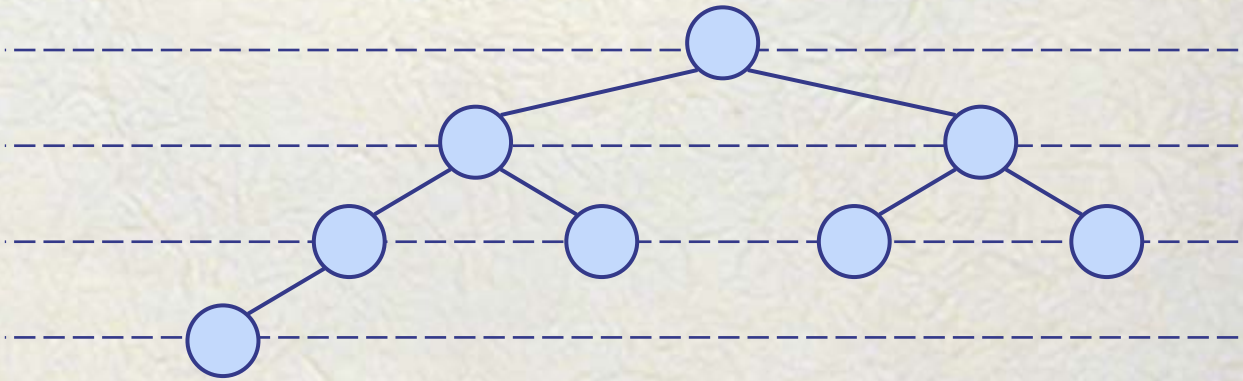


Height of a Heap

- Theorem: A heap storing n keys has height $O(\log n)$

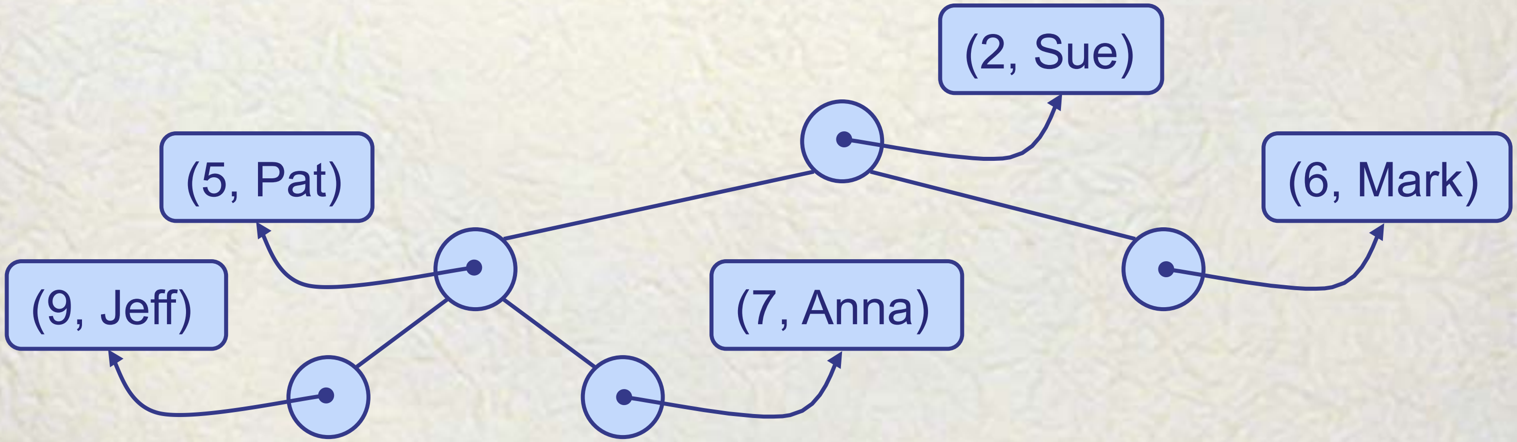
  Proof: (we apply the complete binary tree property)
  - Let h be the height of a heap storing n keys
  - Since there are 2^i keys at depth $i = 0, \ldots, h - 1$ and at least one key at depth h, we have $n \geq 1 + 2 + 4 + \ldots + 2^{h-1} + 1$
  - Thus, $n \geq 2^h$, i.e., $h \leq \log n$

depth	keys
0	1
1	2
$h-1$	2^{h-1}
h	1



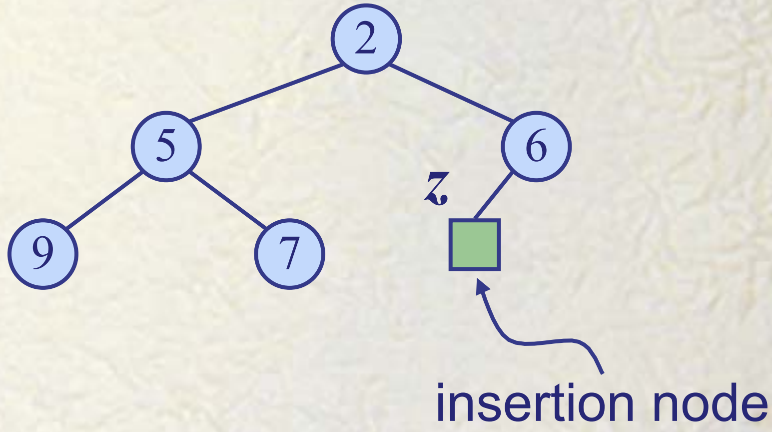
Heaps and Priority Queues

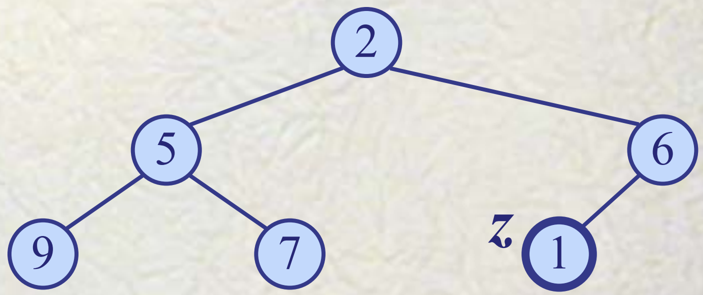
- We can use a heap to implement a priority queue
- We store a (key, element) item at each internal node
- We keep track of the position of the last node
- For simplicity, we show only the keys in the pictures



Insertion into a Heap

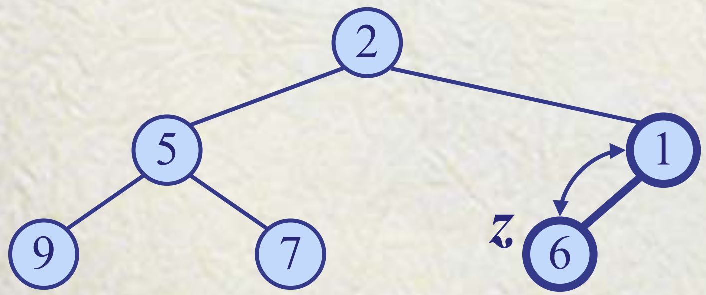
- Method insertItem of the priority queue ADT corresponds to the insertion of a key ***k*** to the heap
- The insertion algorithm consists of three steps
  - Find the insertion node z (the new last node)
  - Store k at z
  - Restore the heap-order property (discussed next)

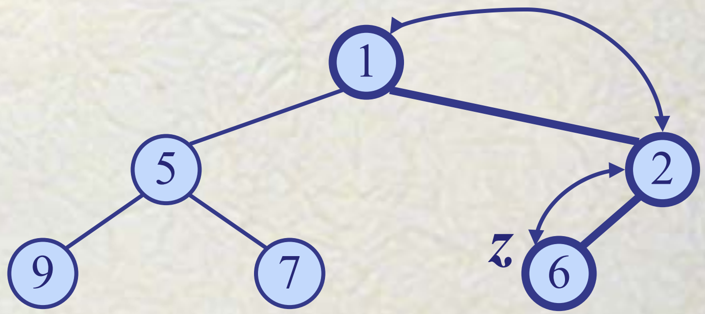




Upheap

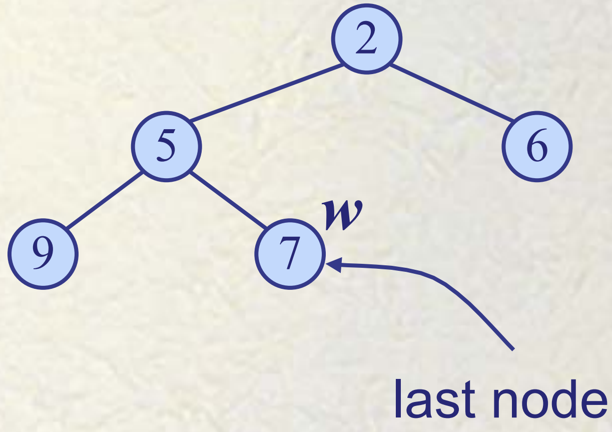
- After the insertion of a new key k, the heap-order property may be violated
- Algorithm upheap restores the heap-order property by swapping k along an upward path from the insertion node
- Upheap terminates when the key k reaches the root or a node whose parent has a key smaller than or equal to k
- Since a heap has height $O(\log n)$, upheap runs in $O(\log n)$ time

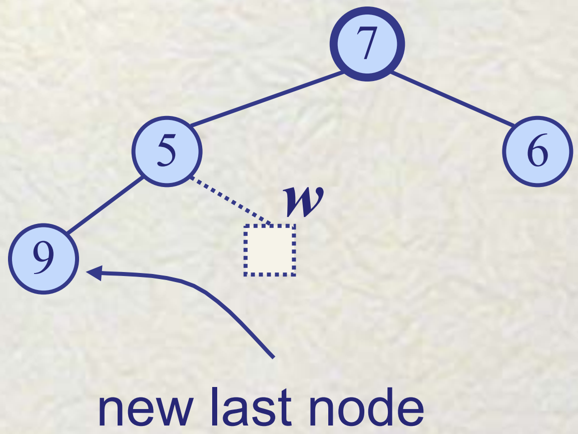




Removal from a Heap

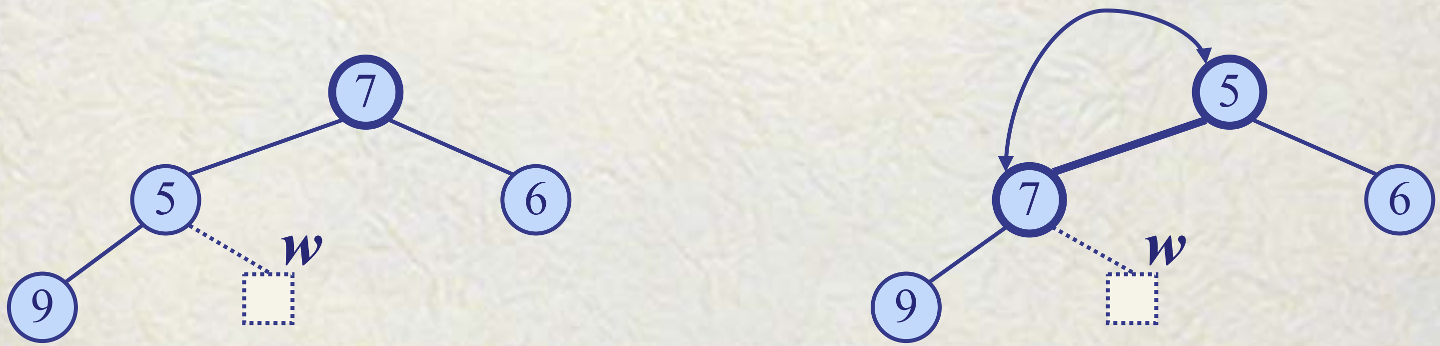
- Method removeMin of the priority queue ADT corresponds to the removal of the root key from the heap
- The removal algorithm consists of three steps
  - Replace the root key with the key of the last node *w*
  - Remove *w*
  - Restore the heap-order property (discussed next)





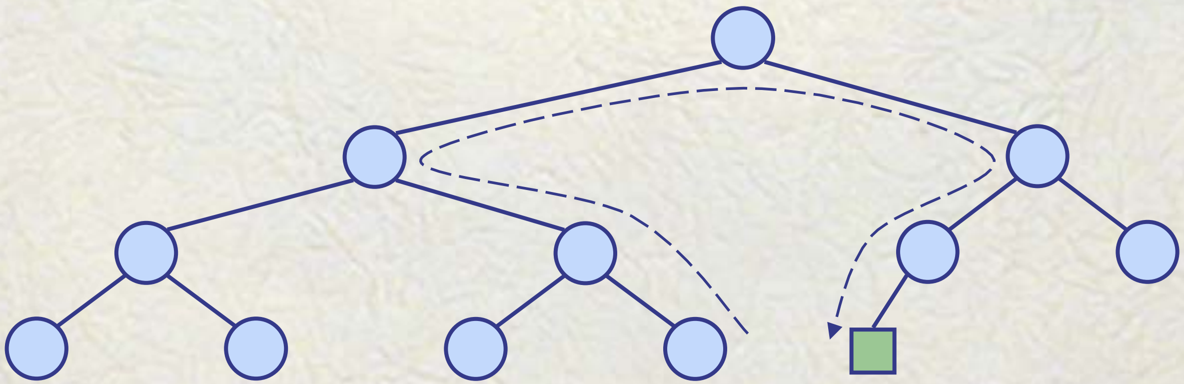
Downheap

- After replacing the root key with the key k of the last node, the heap-order property may be violated
- Algorithm downheap restores the heap-order property by swapping key k along a downward path from the root
- Downheap terminates when key k reaches a leaf or a node whose children have keys greater than or equal to k
- Since a heap has height $O(\log n)$, downheap runs in $O(\log n)$ time



Updating the Last Node

- The insertion node can be found by traversing a path of $O(\log n)$ nodes
  - Go up until a left child or the root is reached
  - If a left child is reached, go to the right child
  - Go down left until a leaf is reached
- Similar algorithm for updating the last node after a removal

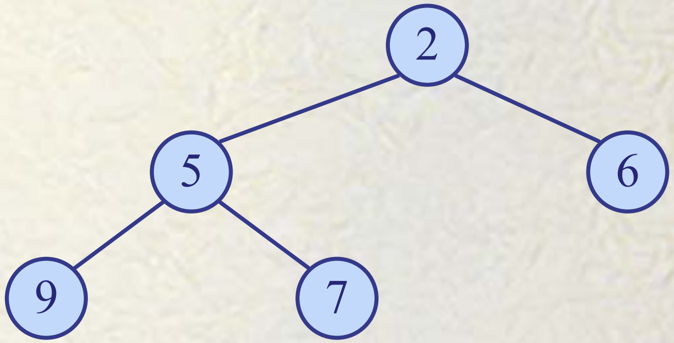


Heap-Sort

- Consider a priority queue with n items implemented by means of a heap
  - the space used is $O(n)$
  - methods insert and removeMin take $O(\log n)$ time
  - methods size, isEmpty, and min take time $O(1)$ time
- Using a heap-based priority queue, we can sort a sequence of n elements in $O(n \log n)$ time
- The resulting algorithm is called heap-sort
- Heap-sort is much faster than quadratic sorting algorithms, such as insertion-sort and selection-sort

Vector-based Heap Implementation

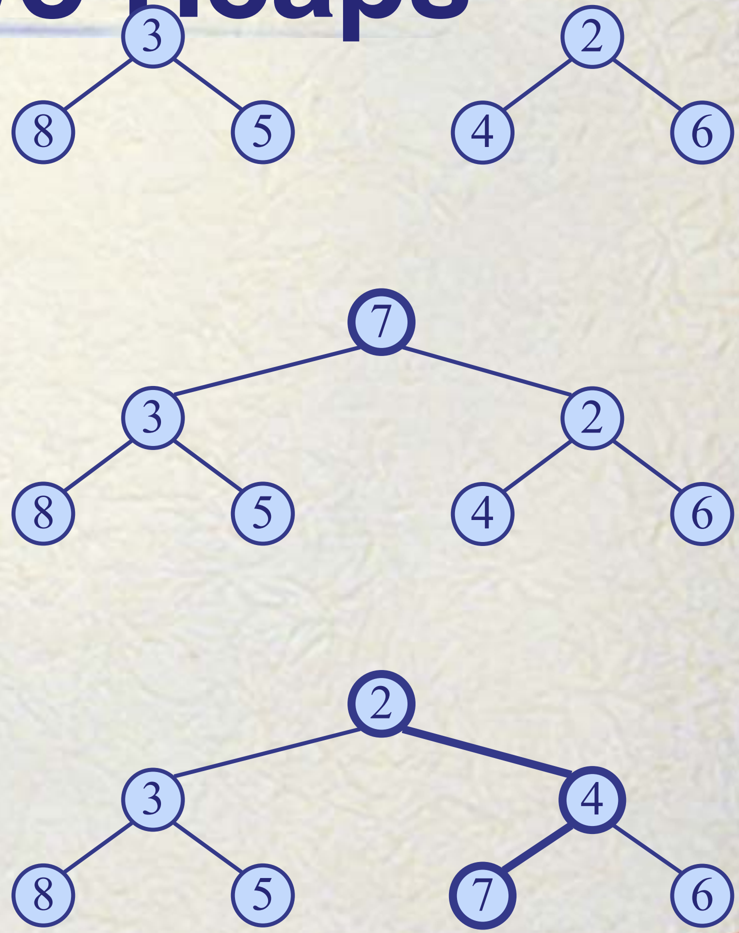
- We can represent a heap with n keys by means of a vector of length $n + 1$
- For the node at rank i
  - the left child is at rank $2i$
  - the right child is at rank $2i + 1$
- Links between nodes are not explicitly stored
- The cell of at rank 0 is not used
- Operation insert corresponds to inserting at rank $n + 1$
- Operation removeMin corresponds to removing at rank 1
- Yields in-place heap-sort



	2	5	6	9	7
0	1	2	3	4	5

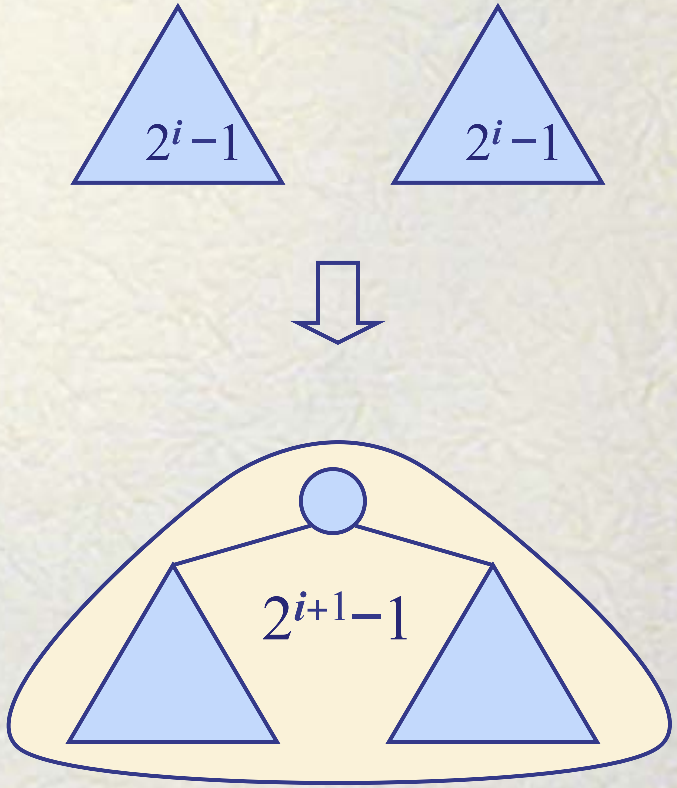
Merging Two Heaps

- We are given two heaps and a key k
- We create a new heap with the root node storing k and with the two heaps as subtrees
- We perform downheap to restore the heap-order property

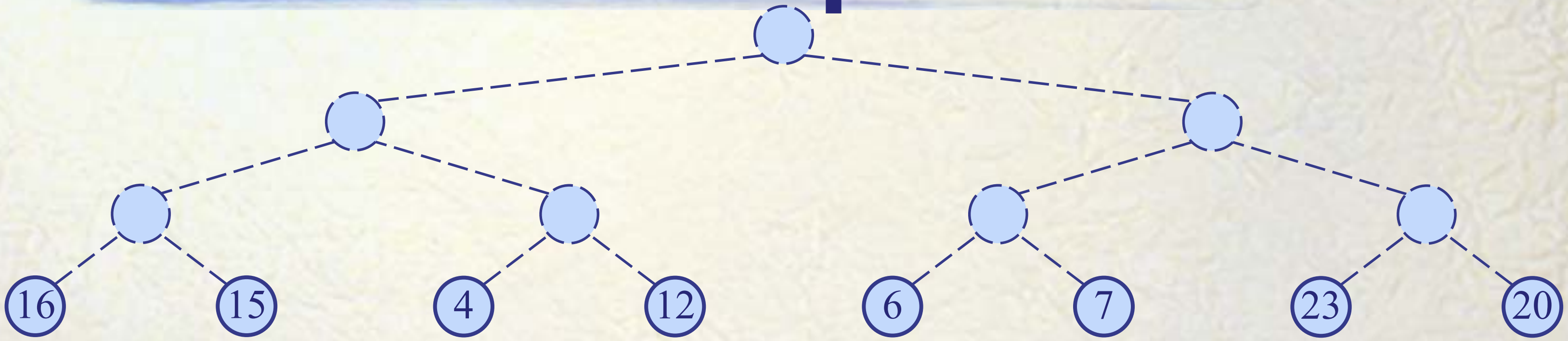


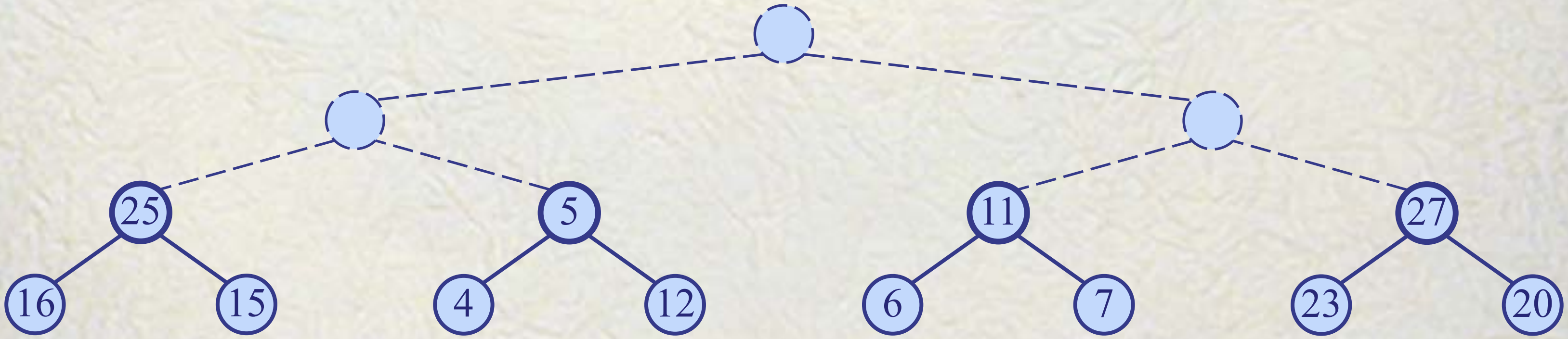
Bottom-up Heap Construction

- We can construct a heap storing n given keys in using a bottom-up construction with $\log n$ phases
- In phase i, pairs of heaps with $2^i - 1$ keys are merged into heaps with $2^{i+1}-1$ keys

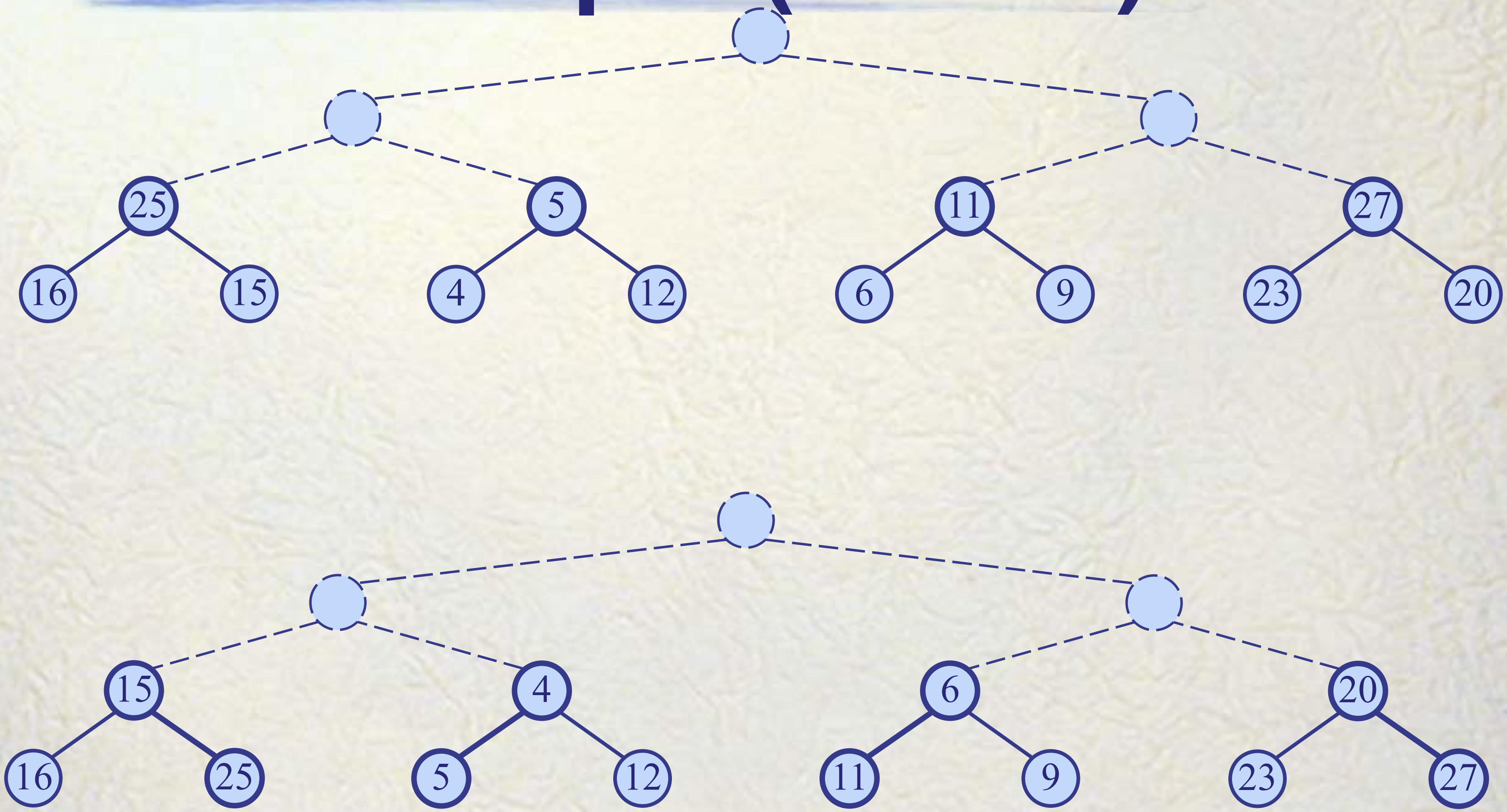


Example

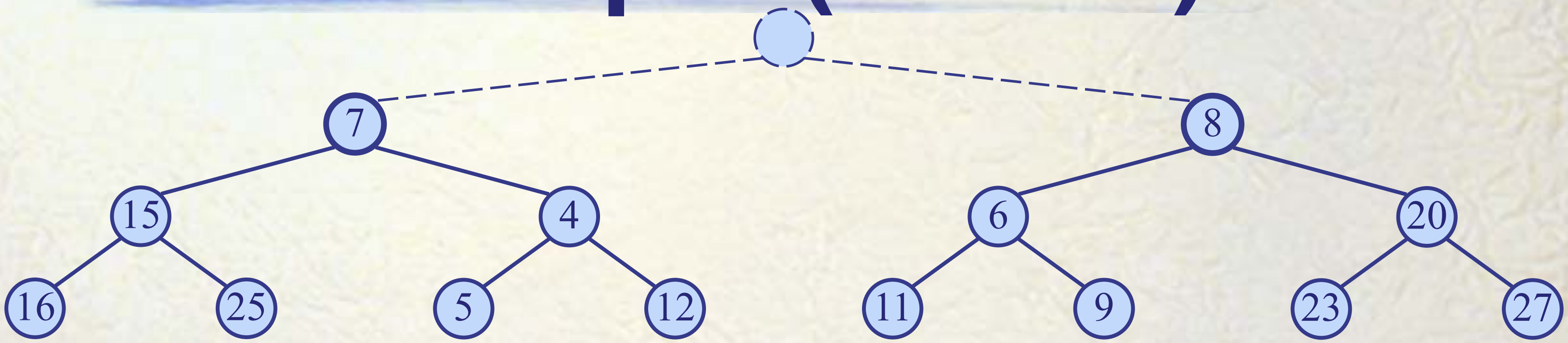


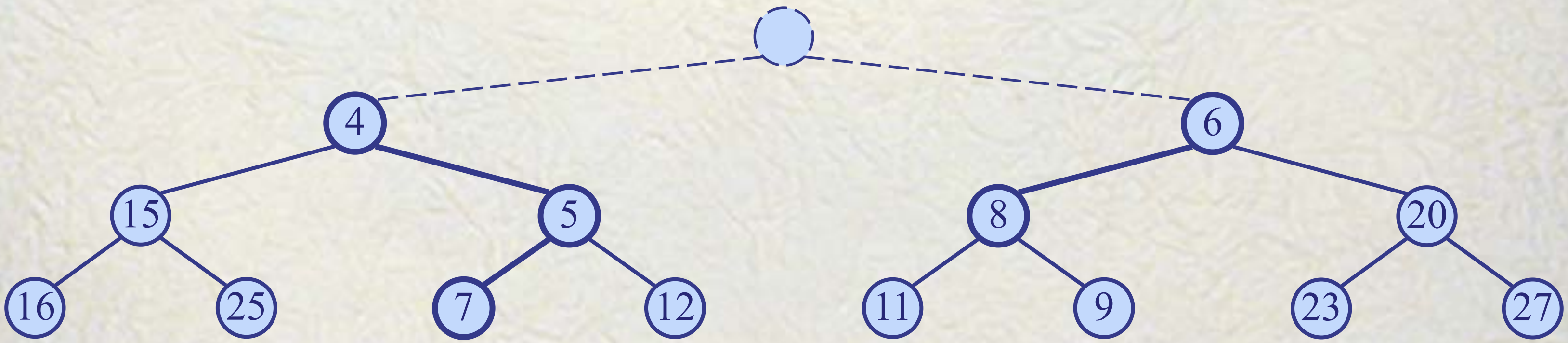


Example (contd.)

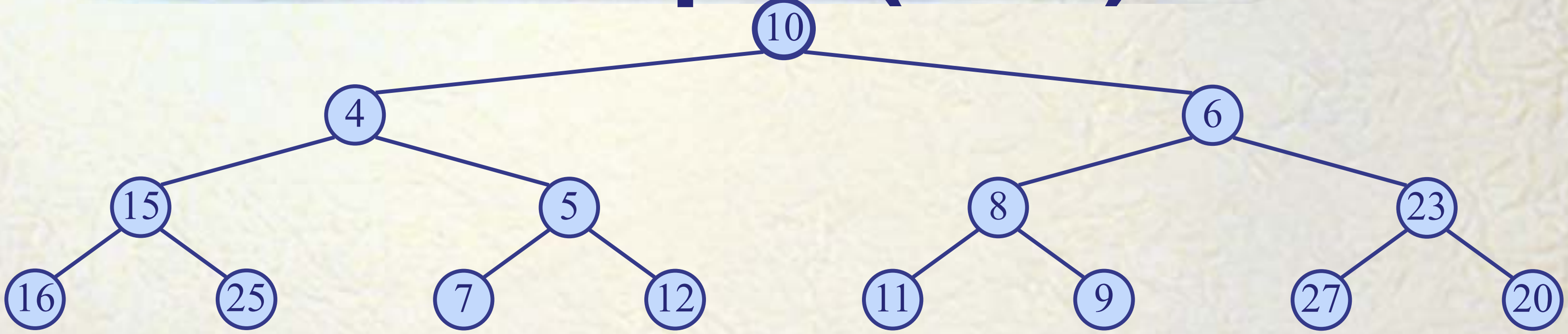


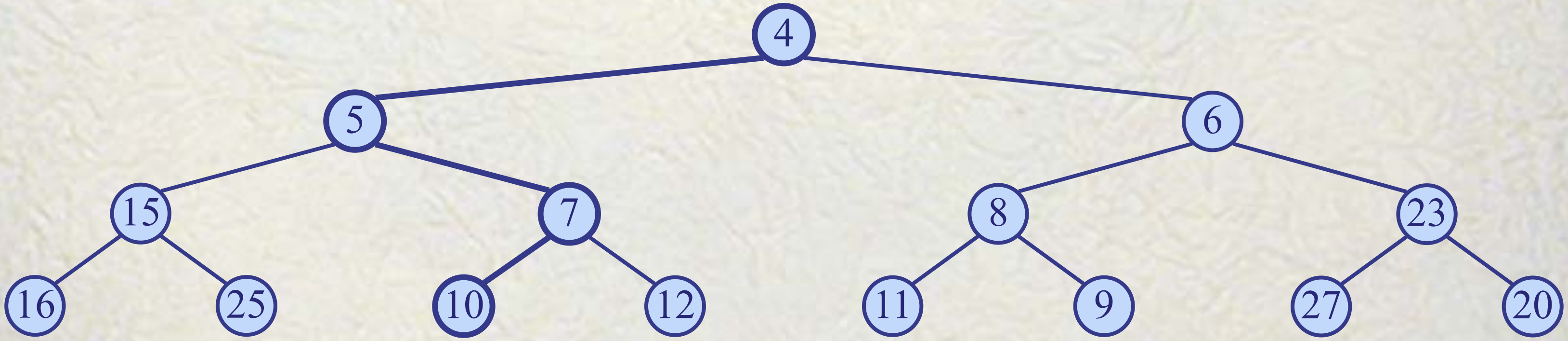
Example (contd.)





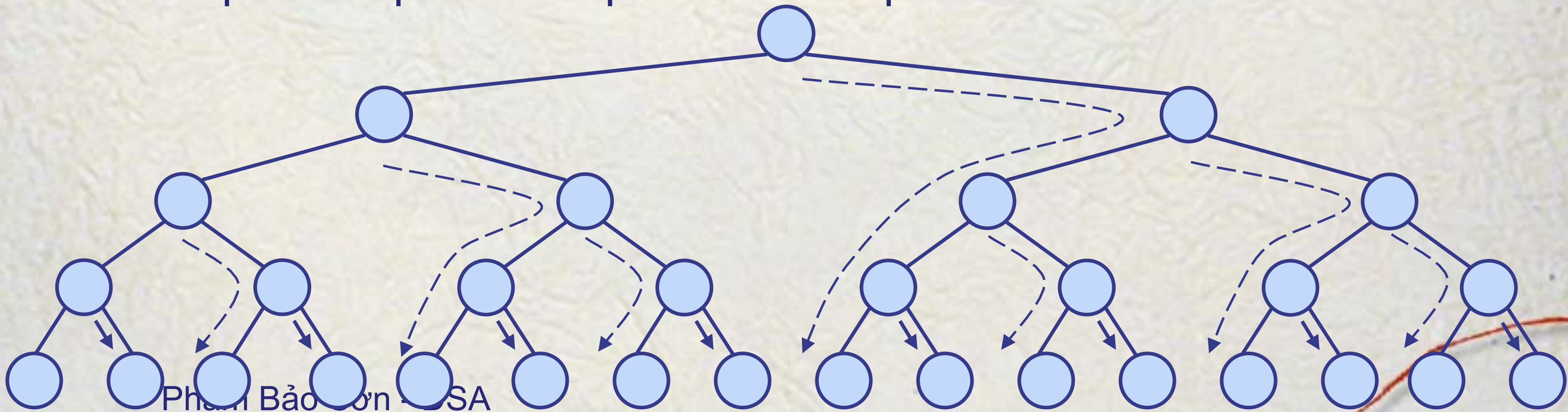
Example (end)



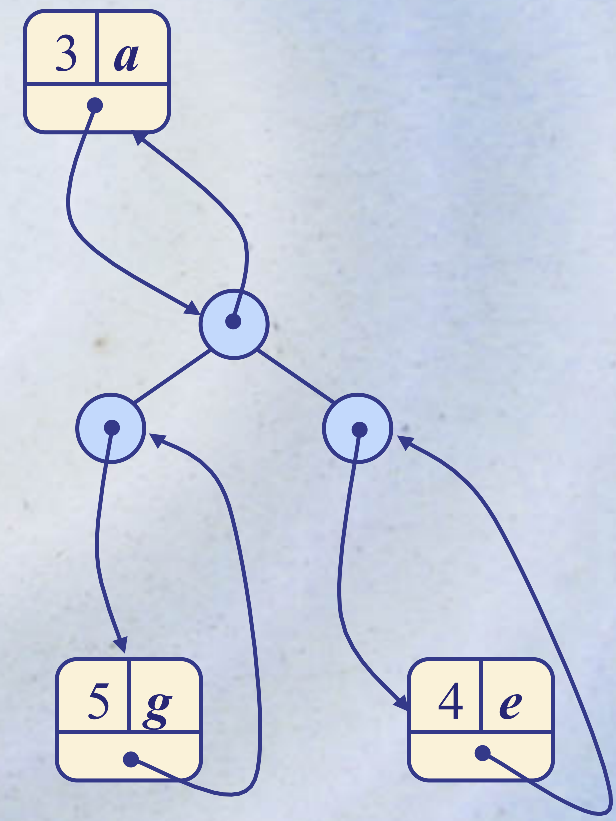


Analysis

- We visualize the worst-case time of a downheap with a proxy path that goes first right and then repeatedly goes left until the bottom of the heap (this path may differ from the actual downheap path)
- Since each node is traversed by at most two proxy paths, the total number of nodes of the proxy paths is $O(n)$
- Thus, bottom-up heap construction runs in $O(n)$ time
- Bottom-up heap construction is faster than n successive insertions and speeds up the first phase of heap-sort



Adaptable Priority Queues



Recall the Entry and Priority Queue ADTs

- An **entry** stores a (key, value) pair within a data structure
- Methods of the entry ADT:
  - key(): returns the key associated with this entry
  - value(): returns the value paired with the key associated with this entry
- Priority Queue ADT:
  - insert(k, x)
    inserts an entry with key k and value x
  - removeMin()
    removes and returns the entry with smallest key
  - min()
    returns, but does not remove, an entry with smallest key
  - size(), isEmpty()

Motivating Example

- Suppose we have an online trading system where orders to purchase and sell a given stock are stored in two priority queues (one for sell orders and one for buy orders) as (p,s) entries:
  - The key, p, of an order is the price
  - The value, s, for an entry is the number of shares
  - A buy order (p,s) is executed when a sell order (p',s') with price p'≤p is added (the execution is complete if s'≥s)
  - A sell order (p,s) is executed when a buy order (p',s') with price p'≥p is added (the execution is complete if s'≥s)
- What if someone wishes to cancel their order before it executes?
- What if someone wishes to update the price or number of shares for their order?

Methods of the Adaptable Priority Queue ADT

- remove(*e*): Remove from *P* and return entry *e*.
- replaceKey(*e,k*): Replace with *k* and return the key of entry *e* of *P;* an error condition occurs if *k* is invalid (that is, *k* cannot be compared with other keys).
- replaceValue(*e,x*): Replace with *x* and return the value of entry *e* of *P*.

Example

Operation	*Output*	P
insert($5,A$)	e_1	$(5,A)$
insert($3,B$)	e_2	$(3,B),(5,A)$
insert($7,C$)	e_3	$(3,B),(5,A),(7,C)$
min()	e_2	$(3,B),(5,A),(7,C)$
key(e_2)	3	$(3,B),(5,A),(7,C)$
remove(e_1)	e_1	$(3,B),(7,C)$
replaceKey($e_2,9$)	3	$(7,C),(9,B)$
replaceValue(e_3,D)	C	$(7,D),(9,B)$
remove(e_2)	e_2	$(7,D)$

Locating Entries

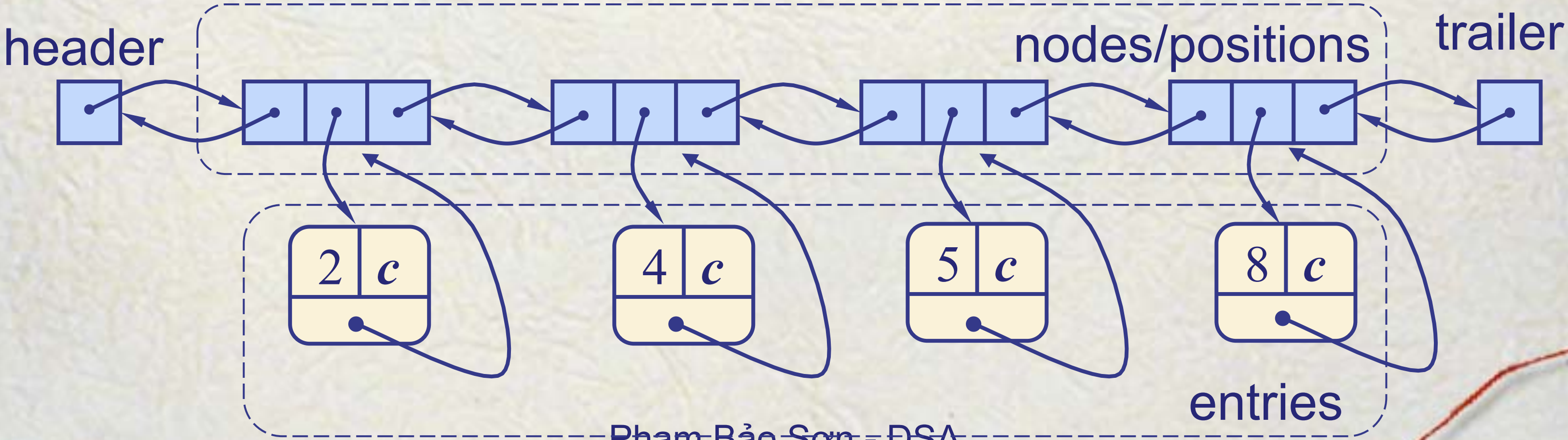
- In order to implement the operations remove(k), replaceKey(e), and replaceValue(k), we need fast ways of locating an entry e in a priority queue.
- We can always just search the entire data structure to find an entry e, but there are better ways for locating entries.

Location-Aware Entries

- A locator-aware entry identifies and tracks the location of its (key, value) object within a data structure
- Intuitive notion:
  - Coat claim check
  - Valet claim ticket
  - Reservation number
- Main idea:
  - Since entries are created and returned from the data structure itself, it can return location-aware entries, thereby making future updates easier

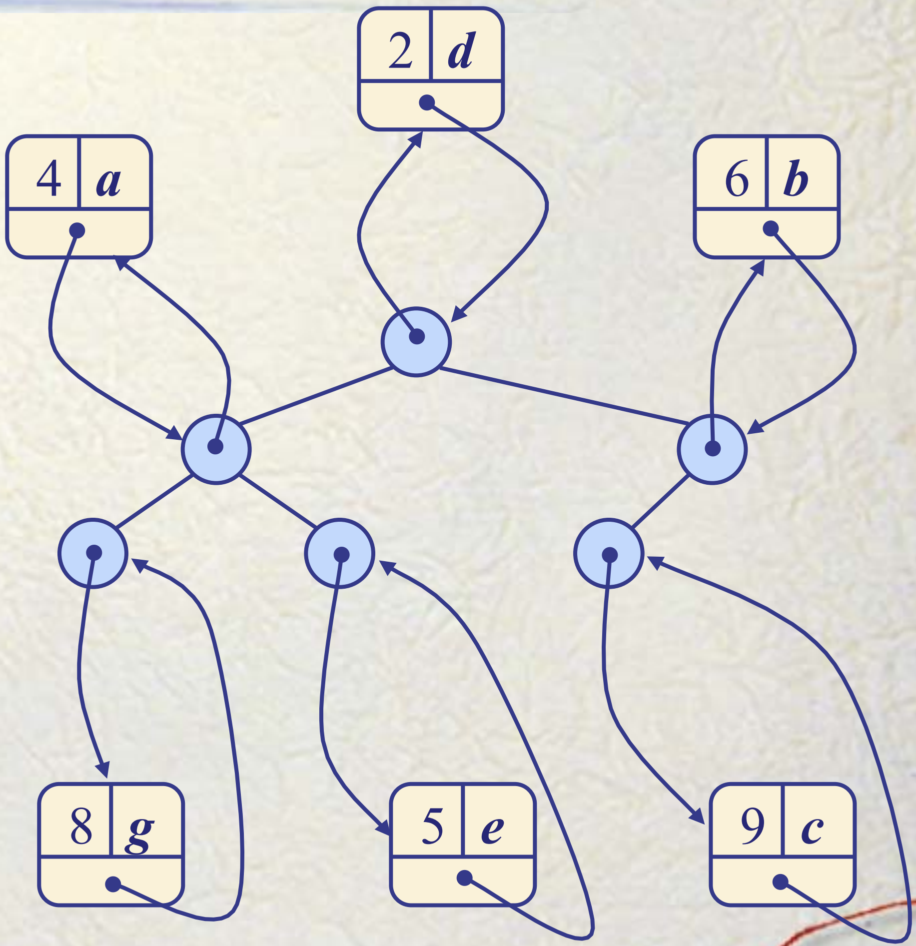
List Implementation

- A location-aware list entry is an object storing
  - key
  - value
  - position (or rank) of the item in the list
- In turn, the position (or array cell) stores the entry
- Back pointers (or ranks) are updated during swaps



Heap Implementation

- A location-aware heap entry is an object storing
  - key
  - value
  - position of the entry in the underlying heap
- In turn, each heap position stores an entry
- Back pointers are updated during entry swaps



Performance

- Using location-aware entries we can achieve the following running times (times better than those achievable without location-aware entries are highlighted in red):

Method	Unsorted List	Sorted List	Heap
size, isEmpty	$O(1)$	$O(1)$	$O(1)$
insert	$O(1)$	$O(n)$	$O(\log n)$
min	$O(n)$	$O(1)$	$O(1)$
removeMin	$O(n)$	$O(1)$	$O(\log n)$
remove	$\boldsymbol{O(1)}$	$\boldsymbol{O(1)}$	$\boldsymbol{O(\log n)}$
replaceKey	$\boldsymbol{O(1)}$	$O(n)$	$\boldsymbol{O(\log n)}$
replaceValue	$\boldsymbol{O(1)}$	$\boldsymbol{O(1)}$	$\boldsymbol{O(1)}$