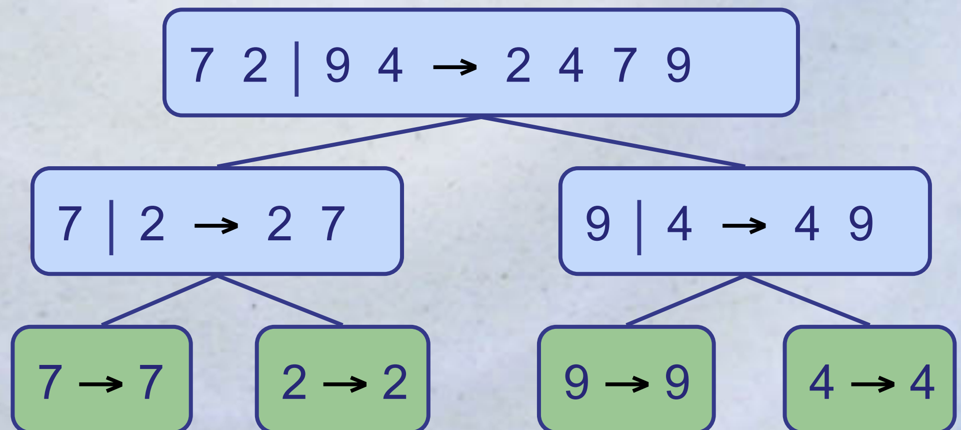
Data Structures and Algorithms

Sorting

Outline

- Merge Sort
- Quick Sort
- Sorting Lower Bound
- Bucket-Sort
- Radix Sort

Merge Sort



Divide-and-Conquer

- Divide-and conquer is a general algorithm design paradigm:
  - Divide: divide the input data S in two disjoint subsets S_1 and S_2
  - Recur: solve the subproblems associated with S_1 and S_2
  - Conquer: combine the solutions for S_1 and S_2 into a solution for S
- The base case for the recursion are subproblems of size 0 or 1
- Merge-sort is a sorting algorithm based on the divide-and-conquer paradigm
- Like heap-sort
  - It uses a comparator
  - It has $O(n \log n)$ running time
- Unlike heap-sort
  - It does not use an auxiliary priority queue
  - It accesses data in a sequential manner (suitable to sort data on a disk)

Merge-Sort

- Merge-sort on an input sequence S with n elements consists of three steps:
  - Divide: partition S into two sequences S_1 and S_2 of about $n/2$ elements each
  - Recur: recursively sort S_1 and S_2
  - Conquer: merge S_1 and S_2 into a unique sorted sequence

Algorithm *mergeSort*(S, C)
    Input sequence S with n elements, comparator C
    Output sequence S sorted according to C
    if $S.size() > 1$
        $(S_1, S_2) \leftarrow$ *partition*(S, $n/2$)
        mergeSort(S_1, C)
        mergeSort(S_2, C)
        $S \leftarrow$ *merge*(S_1, S_2)

Merging Two Sorted Sequences

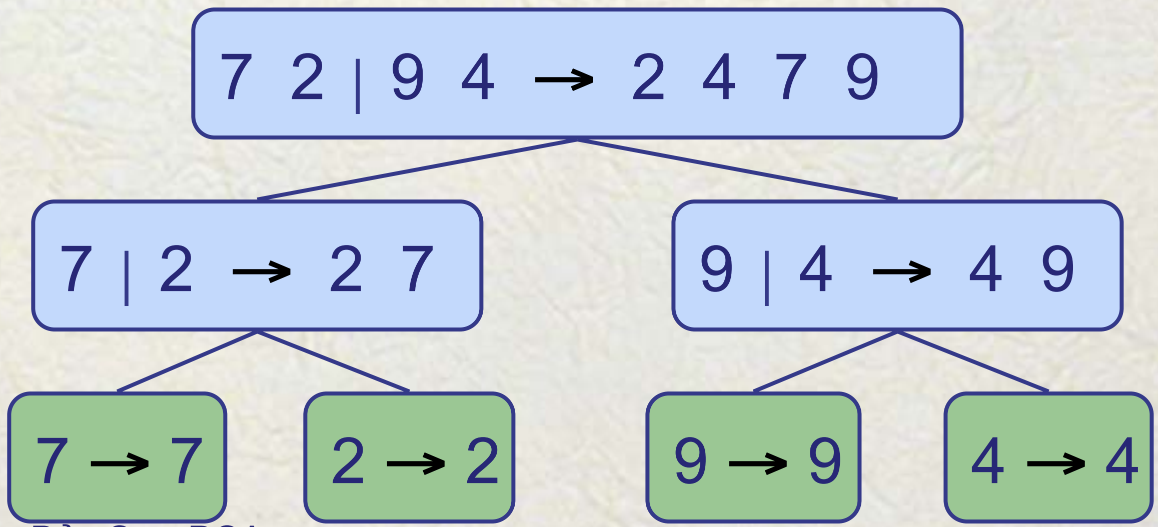
- The conquer step of merge-sort consists of merging two sorted sequences A and B into a sorted sequence S containing the union of the elements of A and B
- Merging two sorted sequences, each with $n/2$ elements and implemented by means of a doubly linked list, takes $O(n)$ time

```
Algorithm merge(A, B)
    Input sequences A and B with
        n/2 elements each
    Output sorted sequence of A ∪ B

    S ← empty sequence
    while ¬A.isEmpty()  ∧ ¬B.isEmpty()
        if A.first().element() < B.first().element()
            S.insertLast(A.remove(A.first()))
        else
            S.insertLast(B.remove(B.first()))
    while ¬A.isEmpty()
        S.insertLast(A.remove(A.first()))
    while ¬B.isEmpty()
        S.insertLast(B.remove(B.first()))
    return S
```

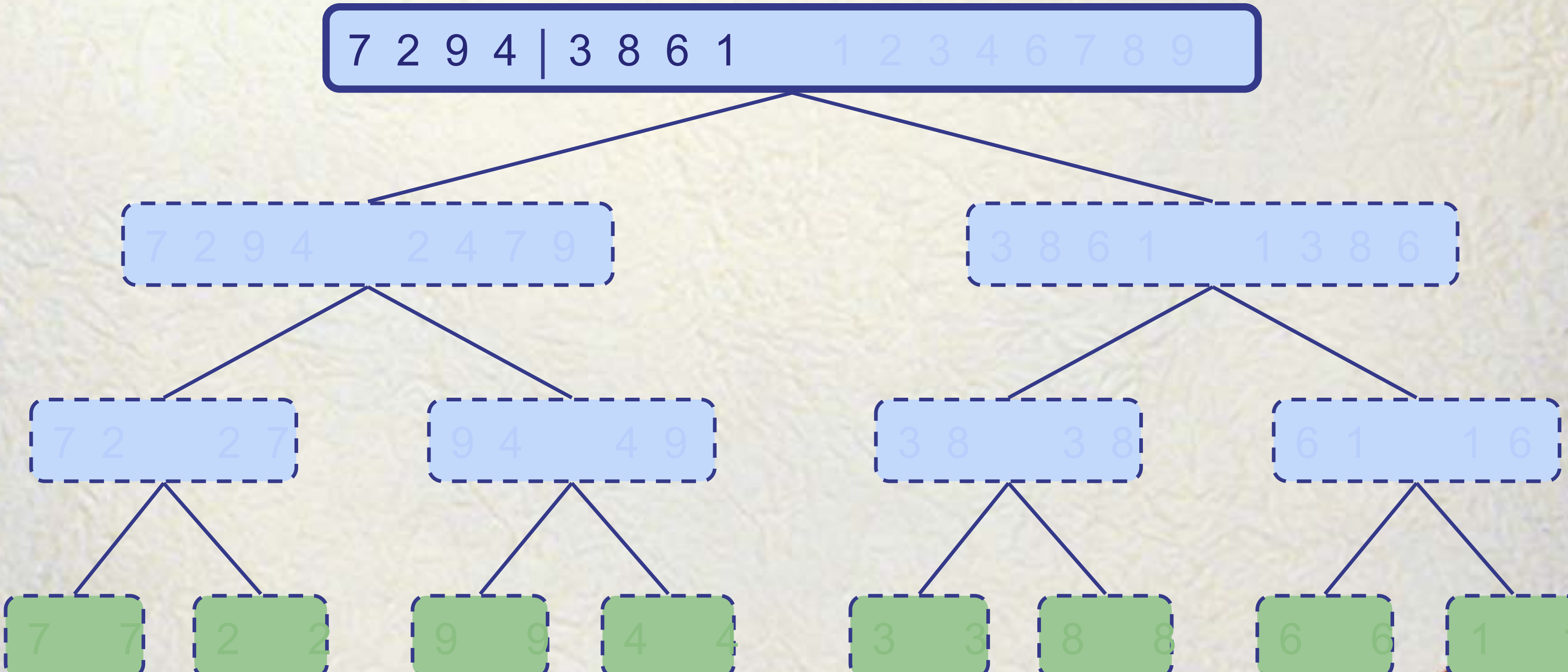
Merge-Sort Tree

- An execution of merge-sort is depicted by a binary tree
  - each node represents a recursive call of merge-sort and stores
    - unsorted sequence before the execution and its partition
    - sorted sequence at the end of the execution
  - the root is the initial call
  - the leaves are calls on subsequences of size 0 or 1



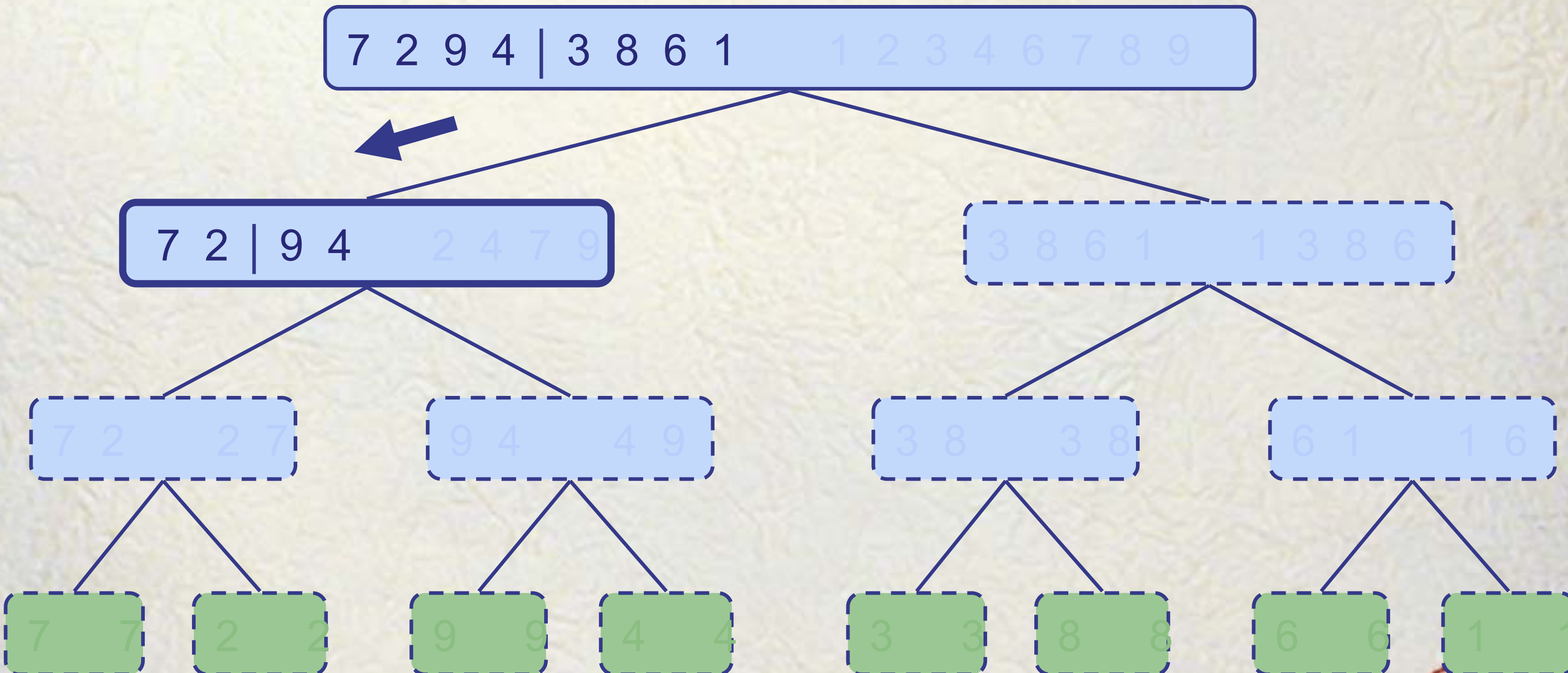
Execution Example

- Partition



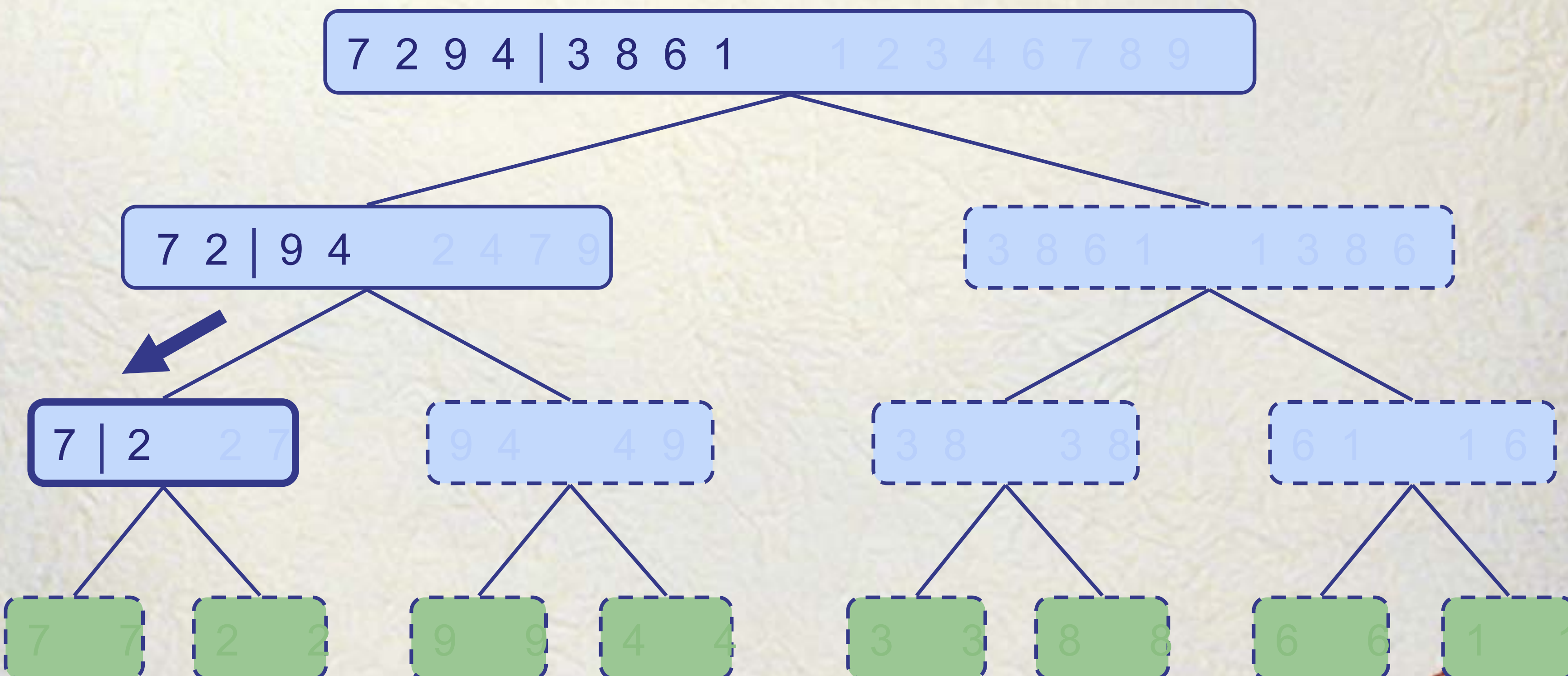
Execution Example (cont.)

- Recursive call, partition



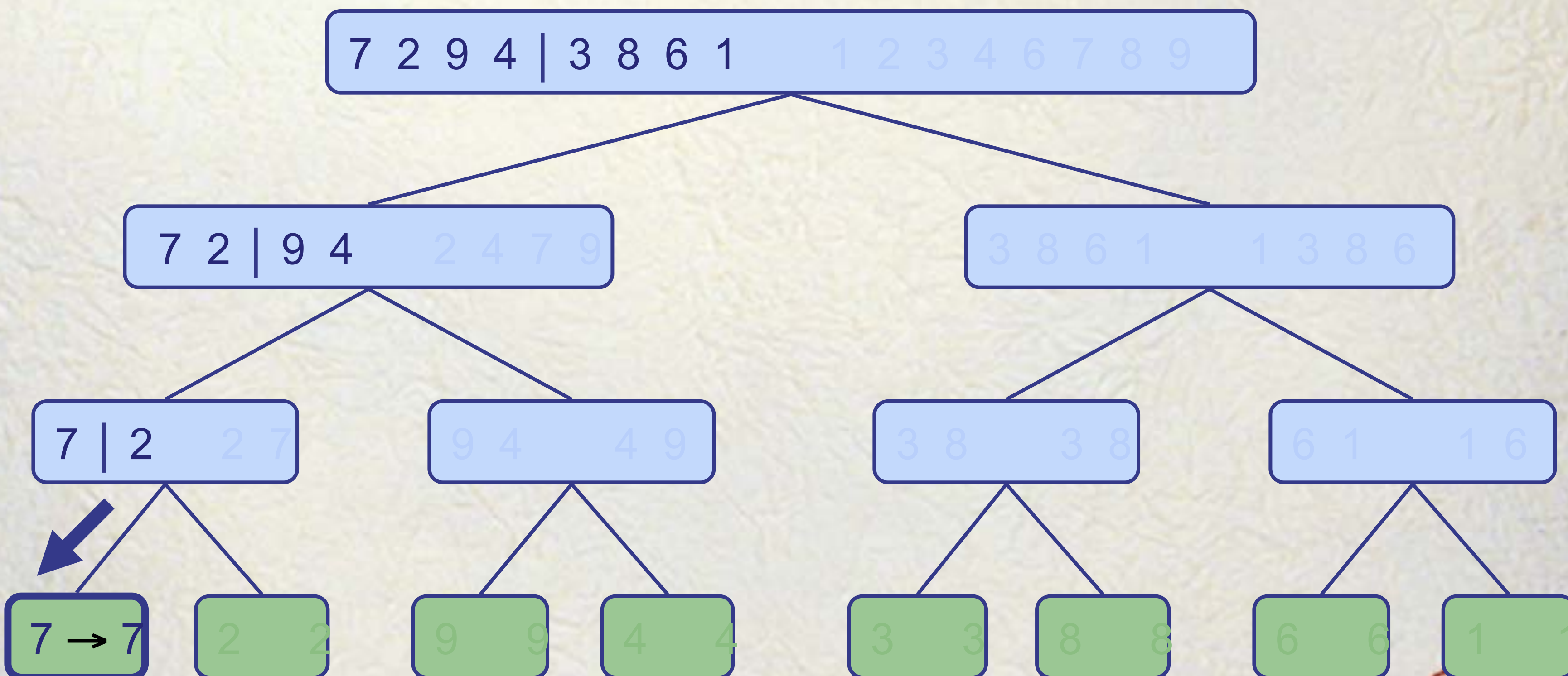
Execution Example (cont.)

- Recursive call, partition



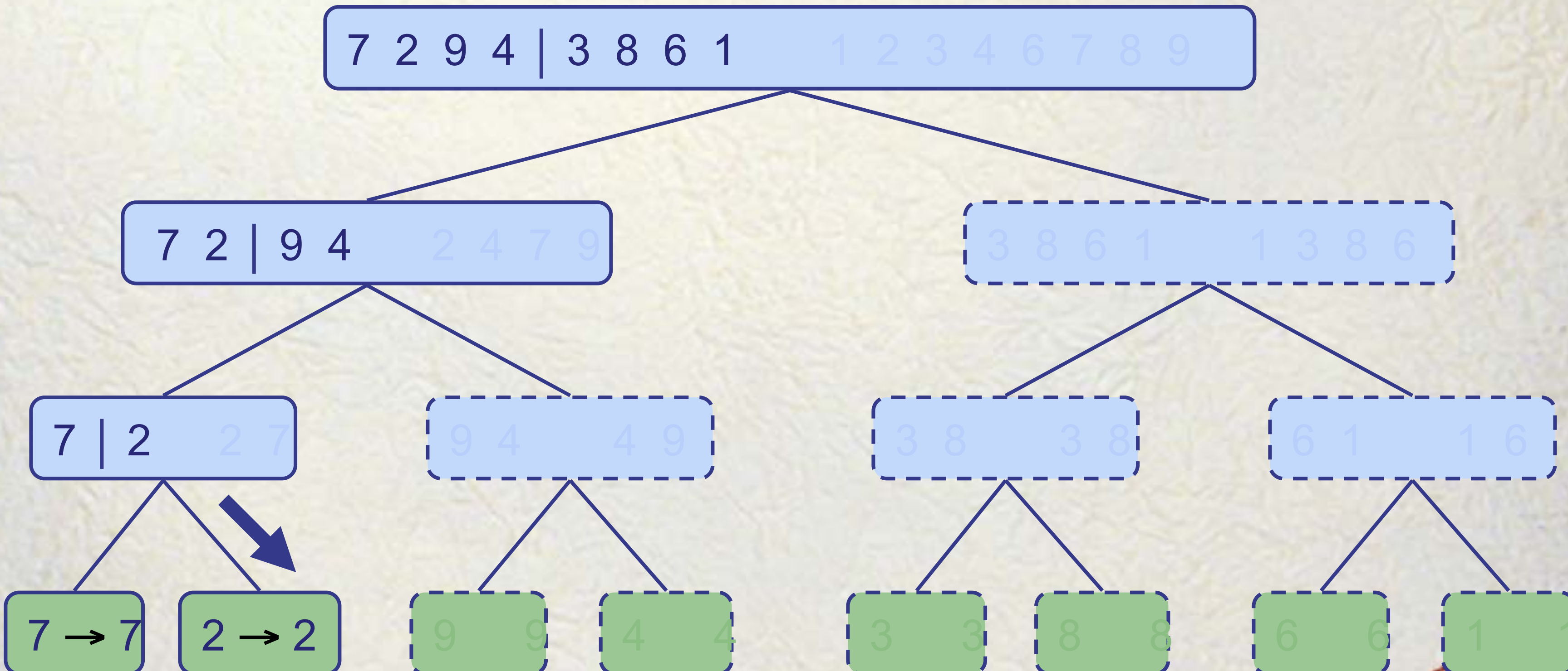
Execution Example (cont.)

- Recursive call, base case



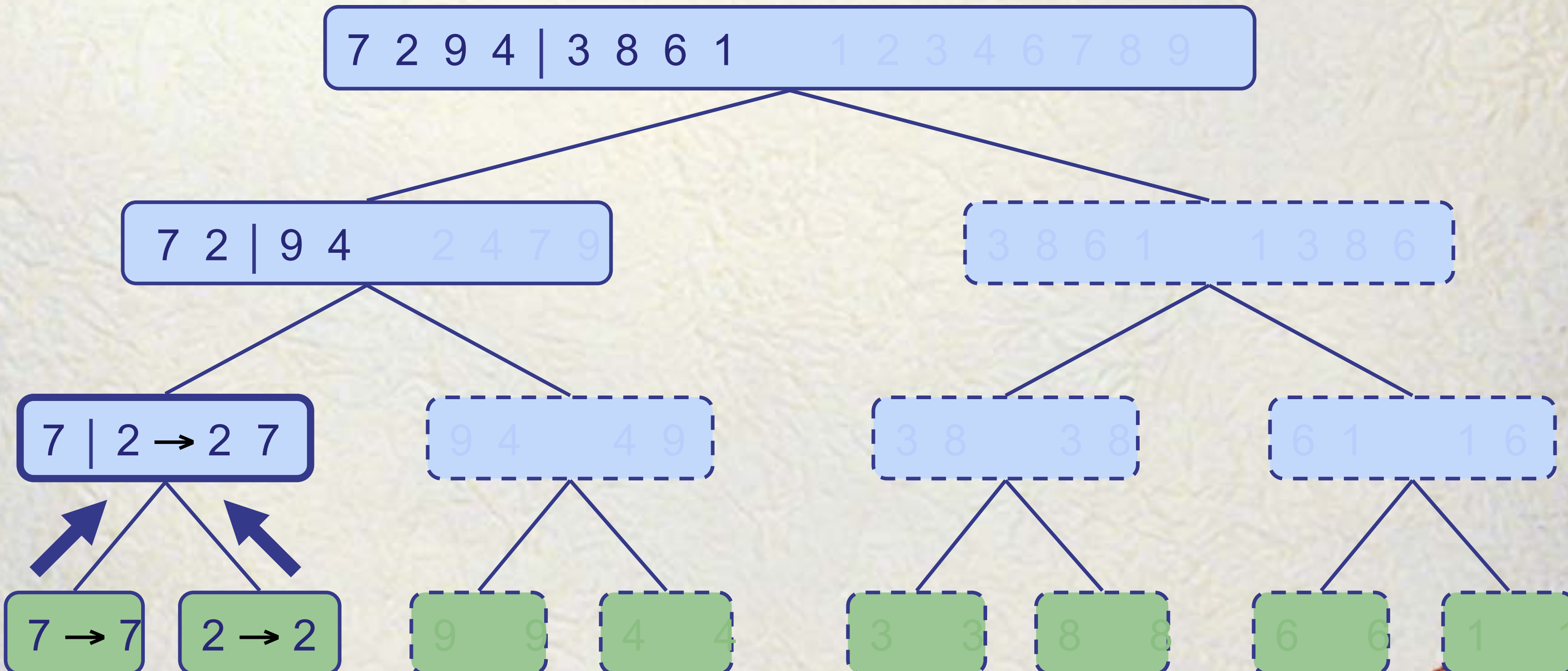
Execution Example (cont.)

- Recursive call, base case



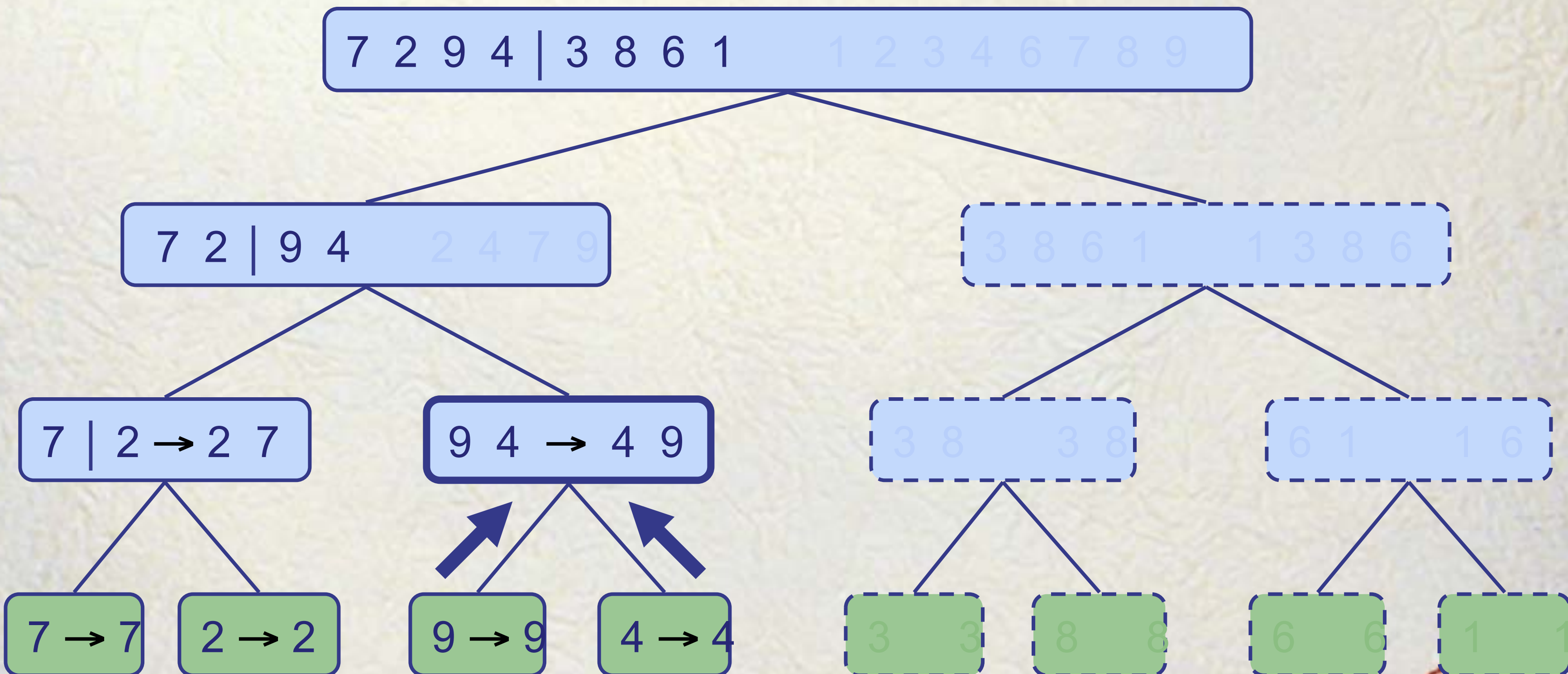
Execution Example (cont.)

- Merge



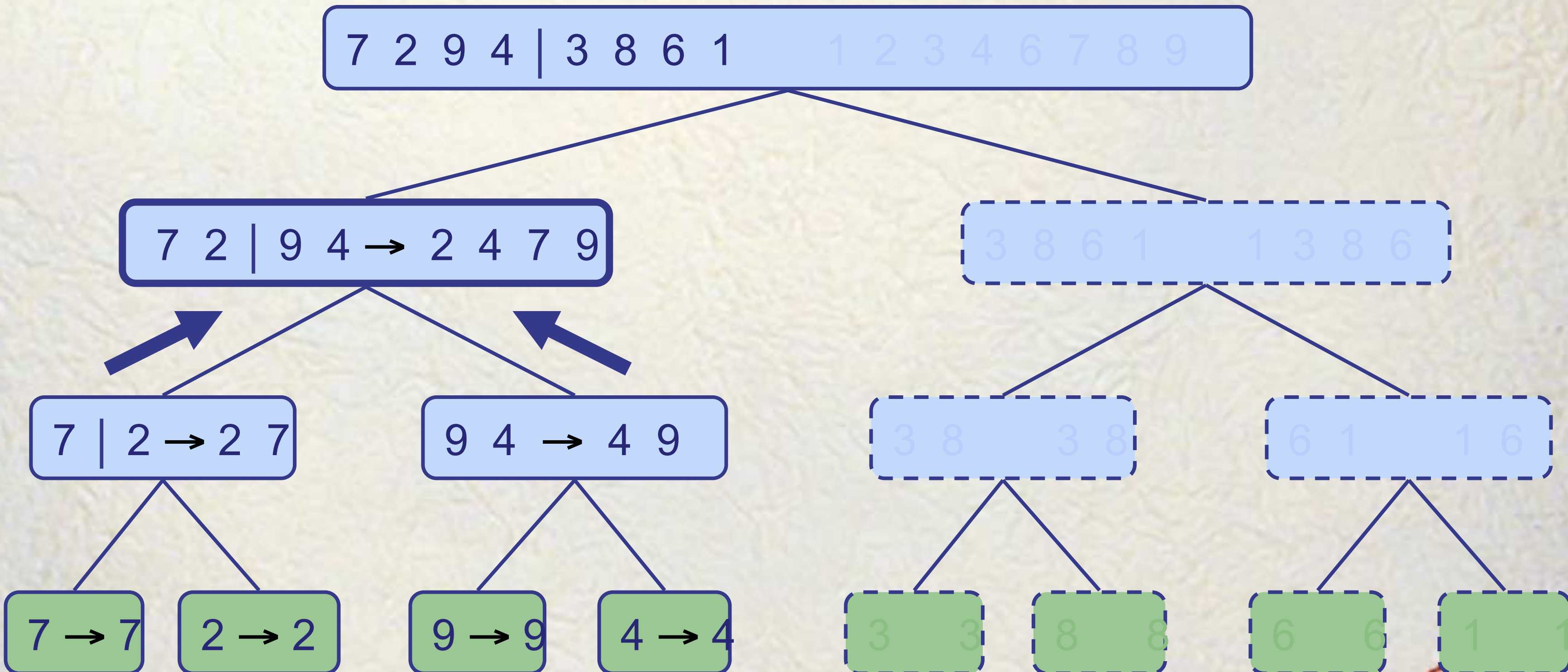
Execution Example (cont.)

- Recursive call, …, base case, merge



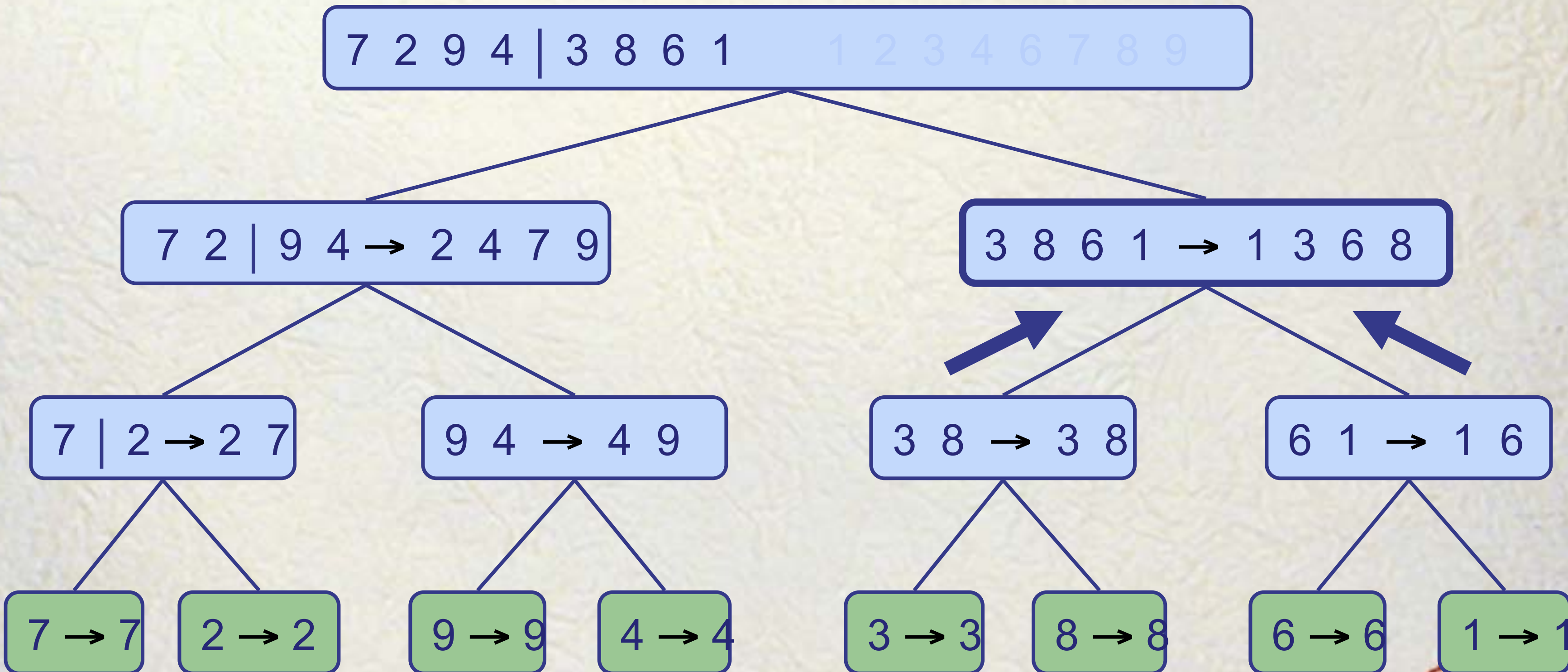
Execution Example (cont.)

- Merge



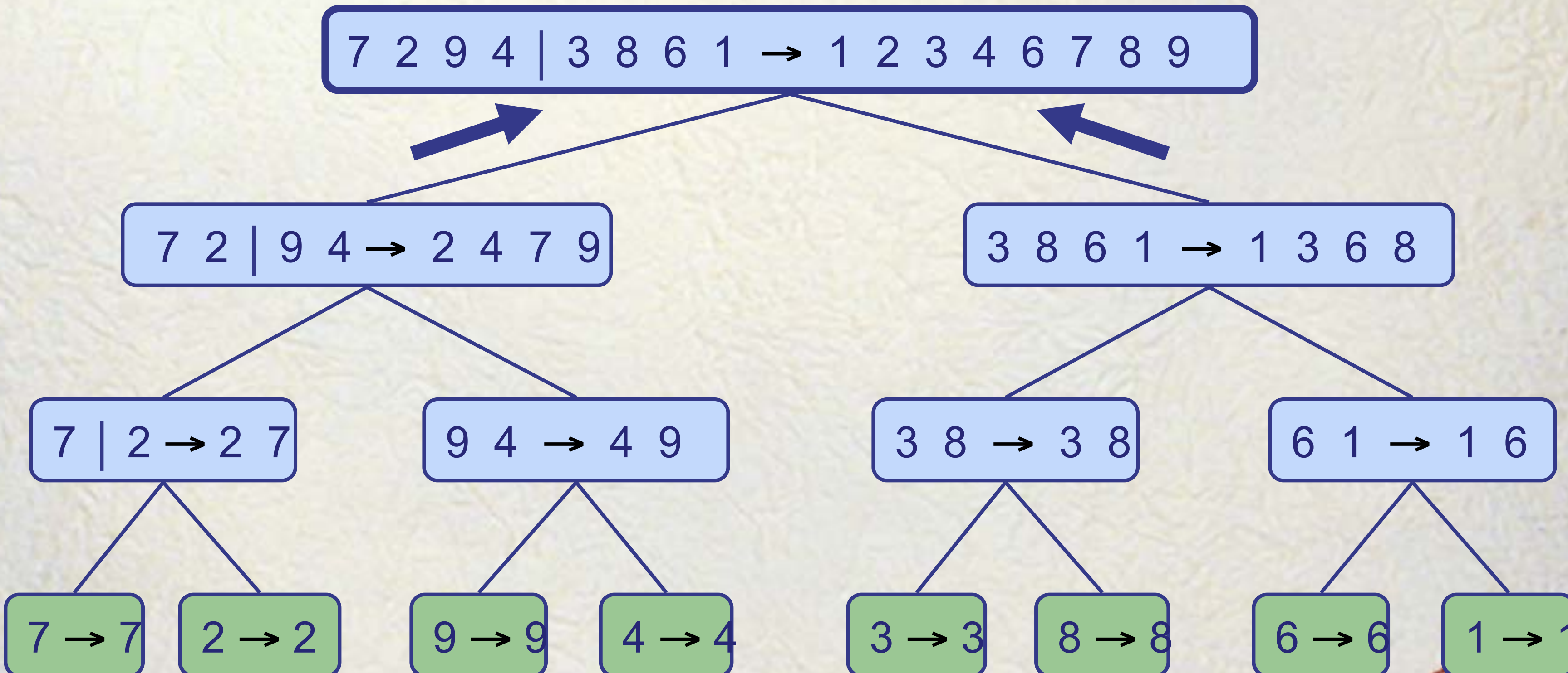
Execution Example (cont.)

- Recursive call, …, merge, merge



Execution Example (cont.)

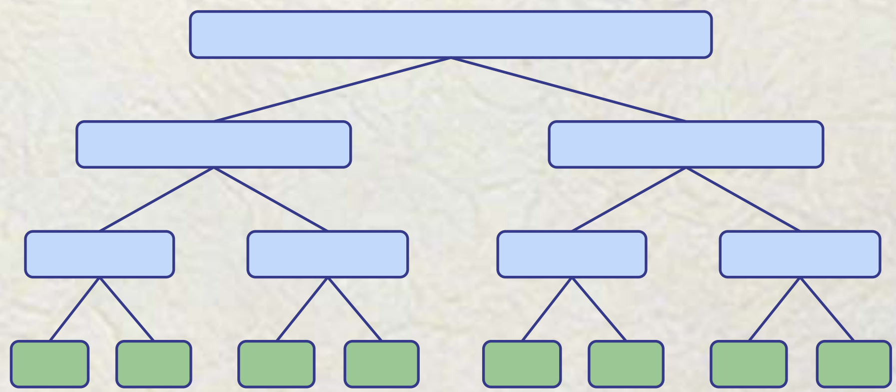
- Merge



Analysis of Merge-Sort

- The height h of the merge-sort tree is $O(\log n)$
  - at each recursive call we divide in half the sequence,
- The overall amount or work done at the nodes of depth i is $O(n)$
  - we partition and merge 2^i sequences of size $n/2^i$
  - we make 2^{i+1} recursive calls
- Thus, the total running time of merge-sort is $O(n \log n)$

depth	#seqs	size
0	1	n
1	2	$n/2$
i	2^i	$n/2^i$
…	…	…



Summary of Sorting Algorithms

Algorithm	Time	Notes
selection-sort	$O(n^2)$	◆ slow ◆ in-place ◆ for small data sets (< 1K)
insertion-sort	$O(n^2)$	◆ slow ◆ in-place ◆ for small data sets (< 1K)
heap-sort	$O(n \log n)$	◆ fast ◆ in-place ◆ for large data sets (1K — 1M)
merge-sort	$O(n \log n)$	◆ fast ◆ sequential data access ◆ for huge data sets (> 1M)

Nonrecursive Merge-Sort

```
public static void mergeSort(Object[] orig, Comparator c) { //
    nonrecursive
  Object[] in = new Object[orig.length]; // make a new temporary array
  System.arraycopy(orig,0,in,0,in.length); // copy the input
  Object[] out = new Object[in.length]; // output array
  Object[] temp; // temp array reference used for swapping
  int n = in.length;
  for (int i=1; i < n; i*=2) { // each iteration sorts all length-2*i runs
    for (int j=0; j < n; j+=2*i)  // each iteration merges two length-i pairs
      merge(in,out,c,j,i); // merge from in to out two length-i runs at j
    temp = in; in = out; out = temp; // swap arrays for next iteration
  }
  // the "in" array contains the sorted array, so re-copy it
  System.arraycopy(in,0,orig,0,in.length);
}
```

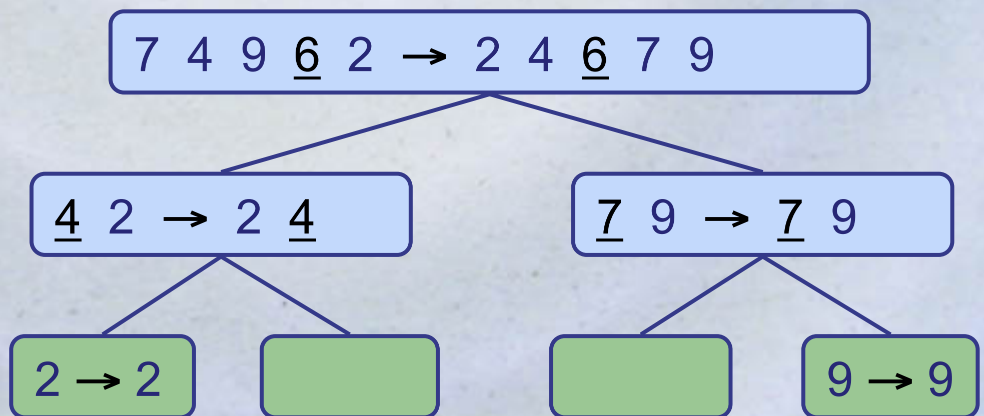
Nonrecursive Merge-Sort

merge runs of length 2, then 4, then 8, and so on

```
public static void mergeSort(Object[] orig, Comparator c) { // nonrecursive
  Object[] in = new Object[orig.length]; // make a new temporary array
  System.arraycopy(orig,0,in,0,in.length); // copy the input
  Object[] out = new Object[in.length]; // output array
  Object[] temp; // temp array reference used for swapping
  int n = in.length;
  for (int i=1; i < n; i*=2) { // each iteration sorts all length-2*i runs
    for (int j=0; j < n; j+=2*i)  // each iteration merges two length-i pairs
      merge(in,out,c,j,i); // merge from in to out two length-i runs at j
    temp = in; in = out; out = temp; // swap arrays for next iteration
  }
  // the "in" array contains the sorted array, so re-copy it
  System.arraycopy(in,0,orig,0,in.length);
}
protected static void merge(Object[] in, Object[] out, Comparator c, int start,
    int inc) { // merge in[start..start+inc-1] and in[start+inc..start+2*inc-1]
  int x = start; // index into run #1
  int end1 = Math.min(start+inc, in.length); // boundary for run #1
  int end2 = Math.min(start+2*inc, in.length); // boundary for run #2
  int y = start+inc; // index into run #2 (could be beyond array boundary)
  int z = start; // index into the out array
  while ((x < end1) && (y < end2))
    if (c.compare(in[x],in[y]) <= 0) out[z++] = in[x++];
    else out[z++] = in[y++];
  if (x < end1) // first run didn't finish
    System.arraycopy(in, x, out, z, end1 - x);
  else if (y < end2) // second run didn't finish
    System.arraycopy(in, y, out, z, end2 - y);
}
```

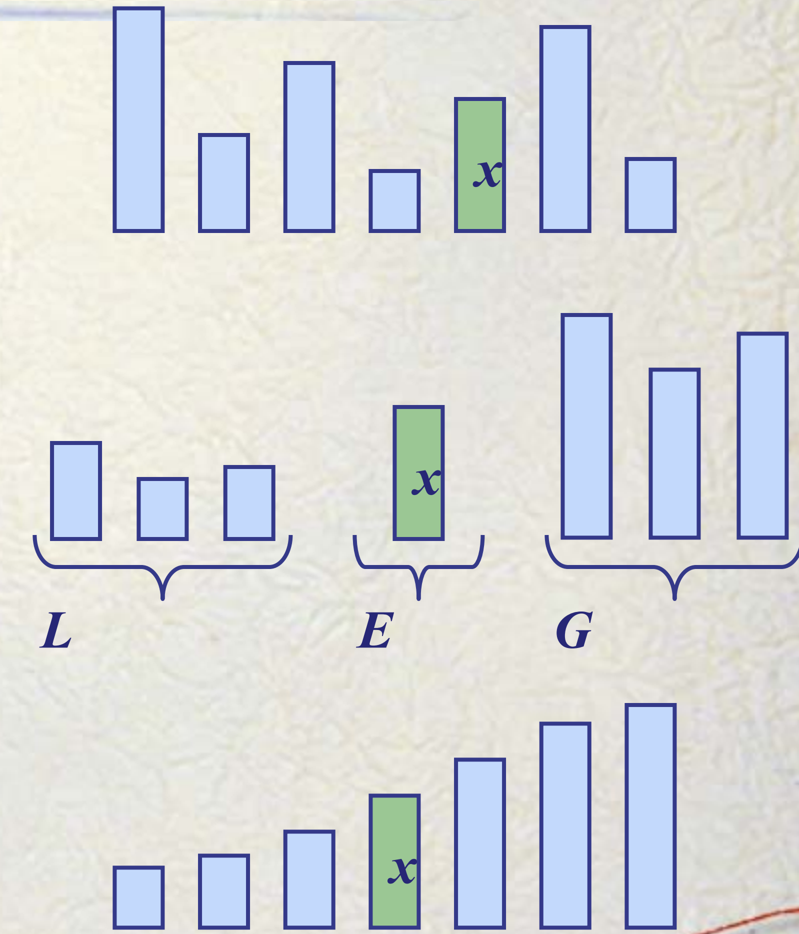
merge two runs in the in array to the out array

Quick-Sort



Quick-Sort

- Quick-sort is a randomized sorting algorithm based on the divide-and-conquer paradigm:
  - Divide: pick a random element x (called pivot) and partition S into
    - L elements less than x
    - E elements equal x
    - G elements greater than x
  - Recur: sort L and G
  - Conquer: join L, E and G



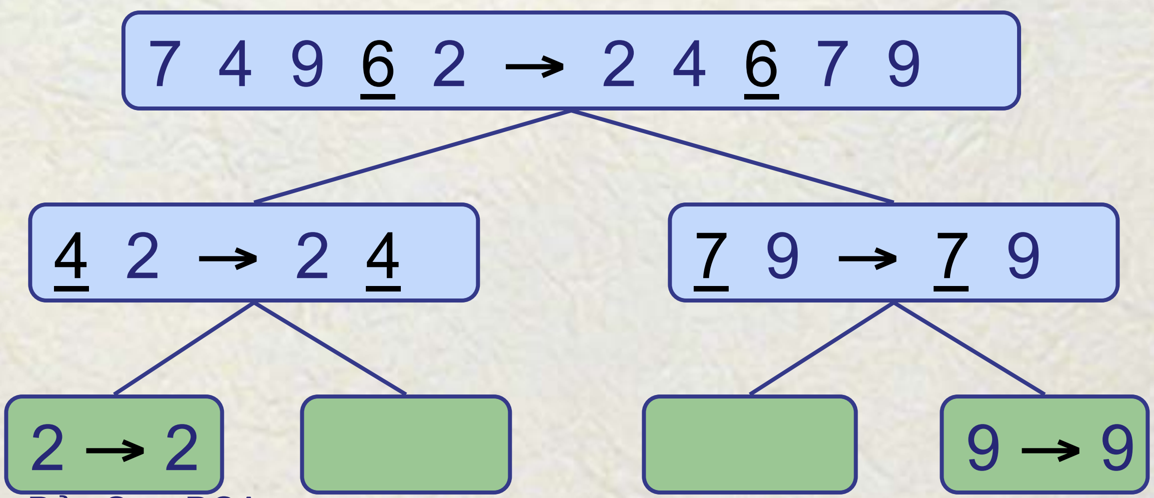
Partition

- We partition an input sequence as follows:
  - We remove, in turn, each element y from S and
  - We insert y into L, E or G, depending on the result of the comparison with the pivot x
- Each insertion and removal is at the beginning or at the end of a sequence, and hence takes $O(1)$ time
- Thus, the partition step of quick-sort takes $O(n)$ time

Algorithm *partition*(S, p)
    Input sequence S, position p of pivot
    Output subsequences L, E, G of the elements of S less than, equal to, or greater than the pivot, resp.
    $L, E, G \leftarrow$ empty sequences
    $x \leftarrow S.remove(p)$
    while $\neg S.isEmpty()$
        $y \leftarrow S.remove(S.first())$
        if $y < x$
            $L.insertLast(y)$
        else if $y = x$
            $E.insertLast(y)$
        else $\{ y > x \}$
            $G.insertLast(y)$
    return L, E, G

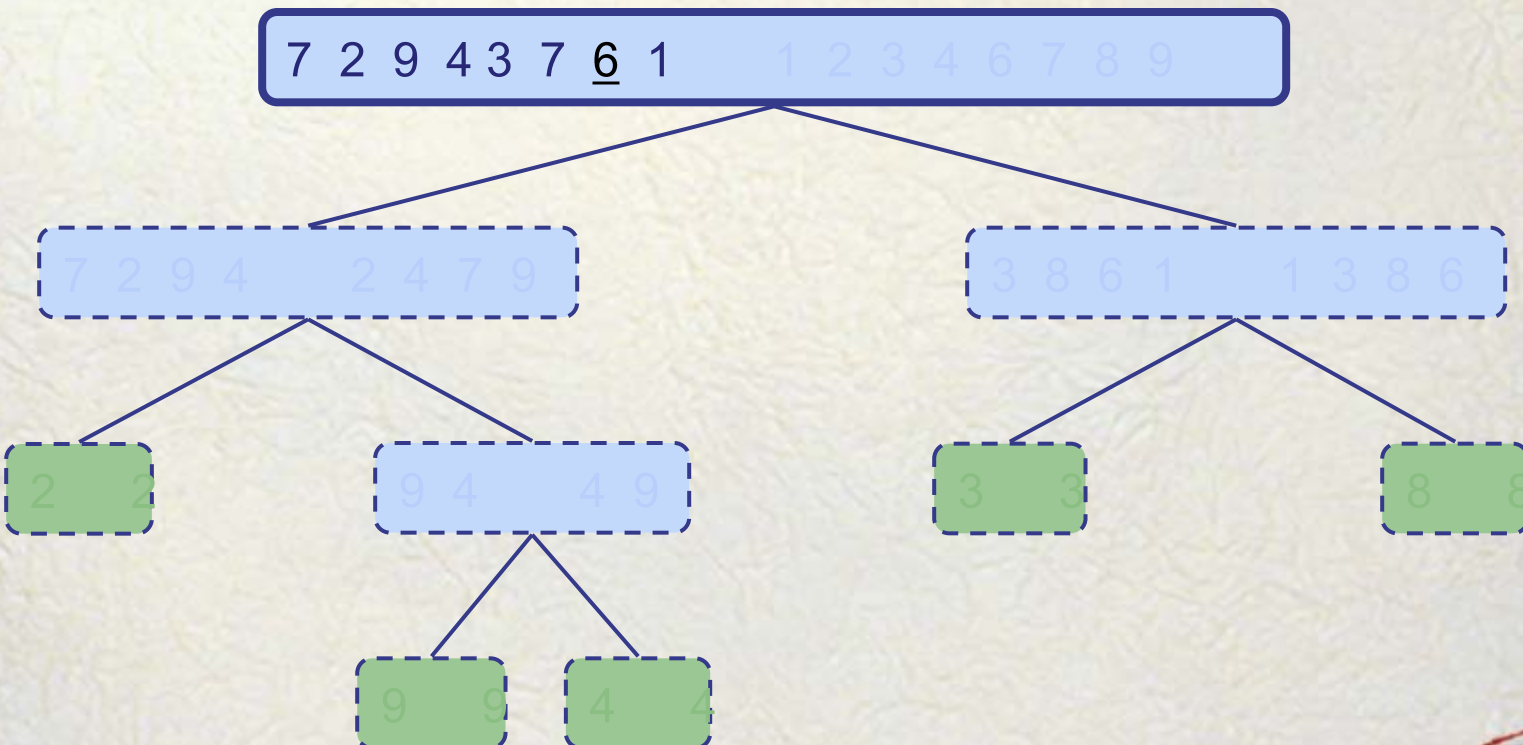
Quick-Sort Tree

- An execution of quick-sort is depicted by a binary tree
  - Each node represents a recursive call of quick-sort and stores
    - Unsorted sequence before the execution and its pivot
    - Sorted sequence at the end of the execution
  - The root is the initial call
  - The leaves are calls on subsequences of size 0 or 1



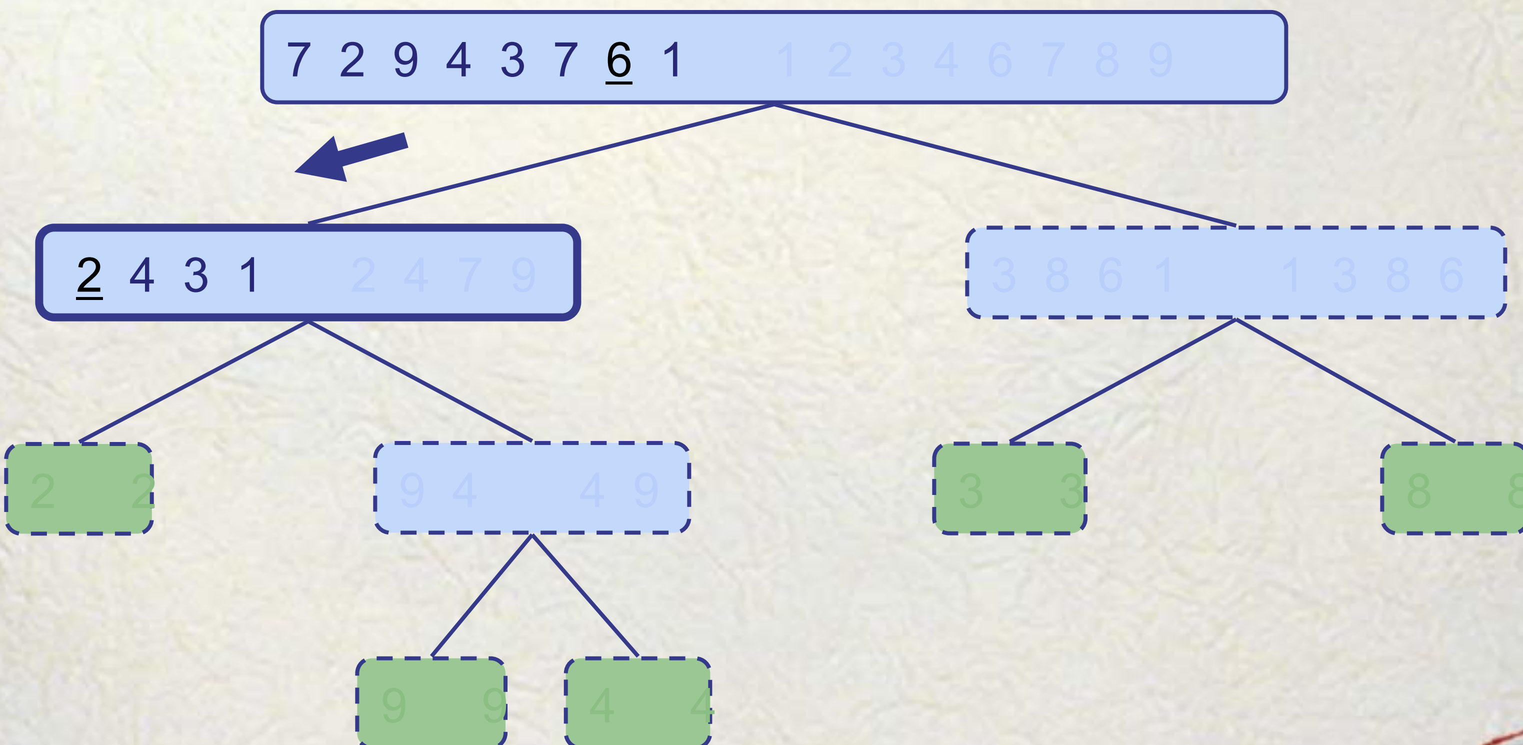
Execution Example

- Pivot selection



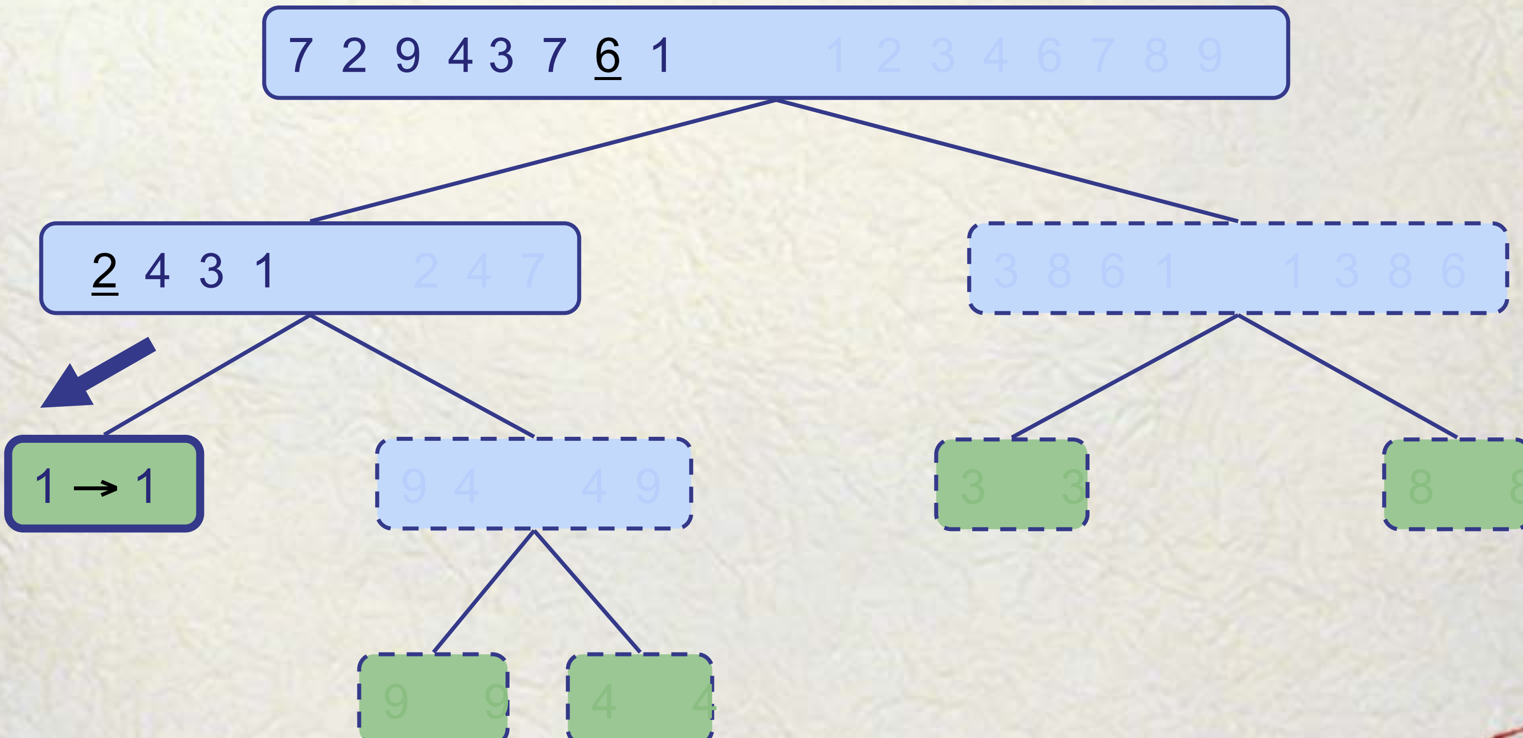
Execution Example (cont.)

- Partition, recursive call, pivot selection



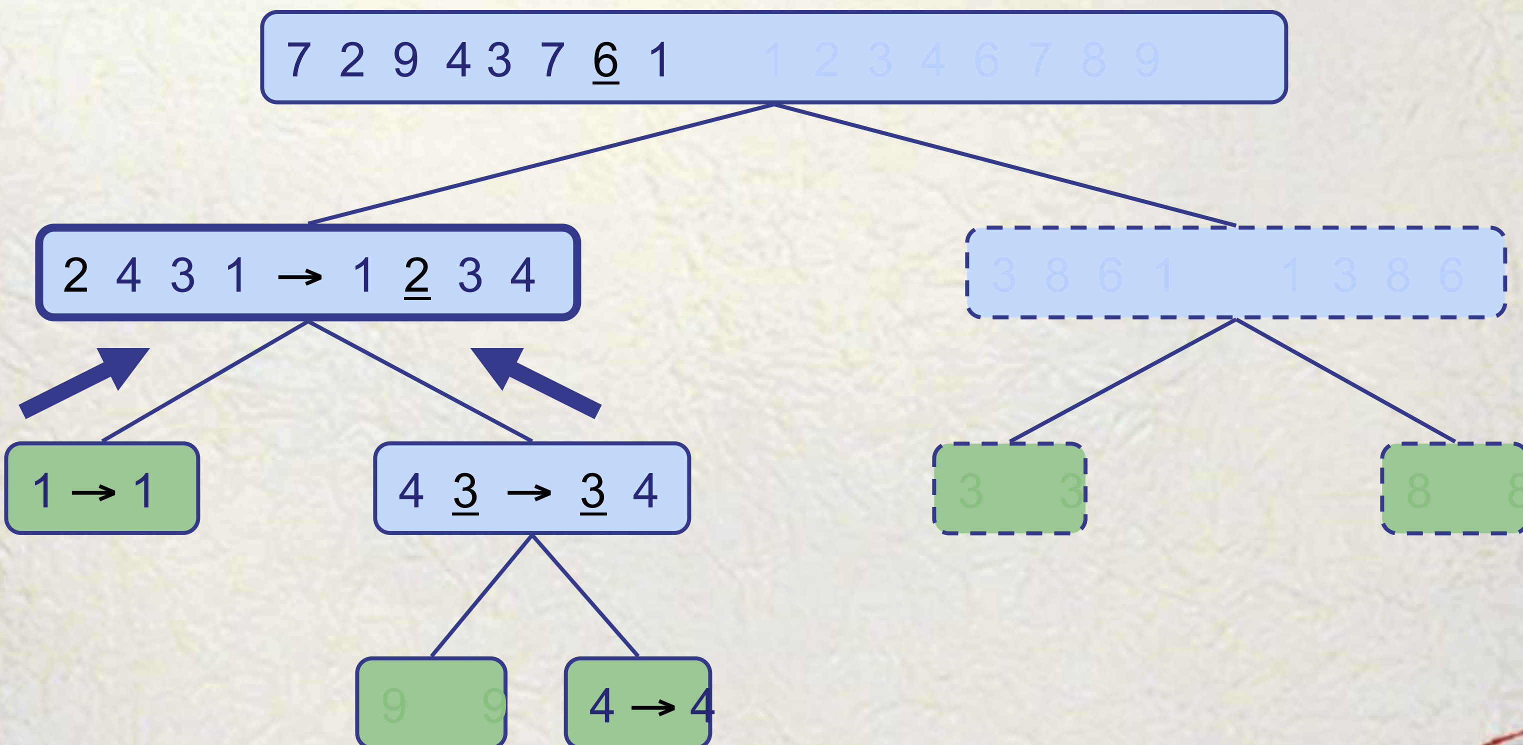
Execution Example (cont.)

- Partition, recursive call, base case



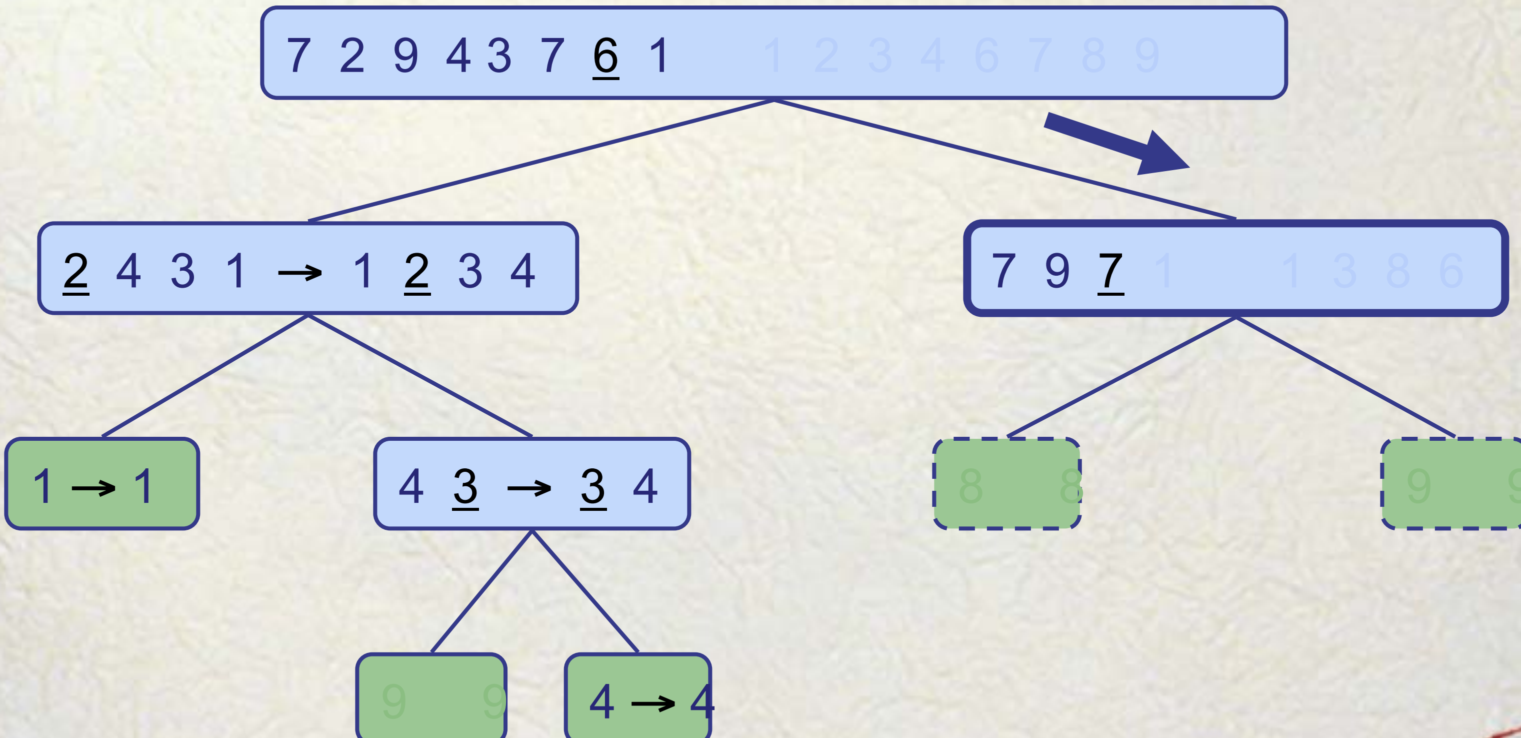
Execution Example (cont.)

- Recursive call, …, base case, join



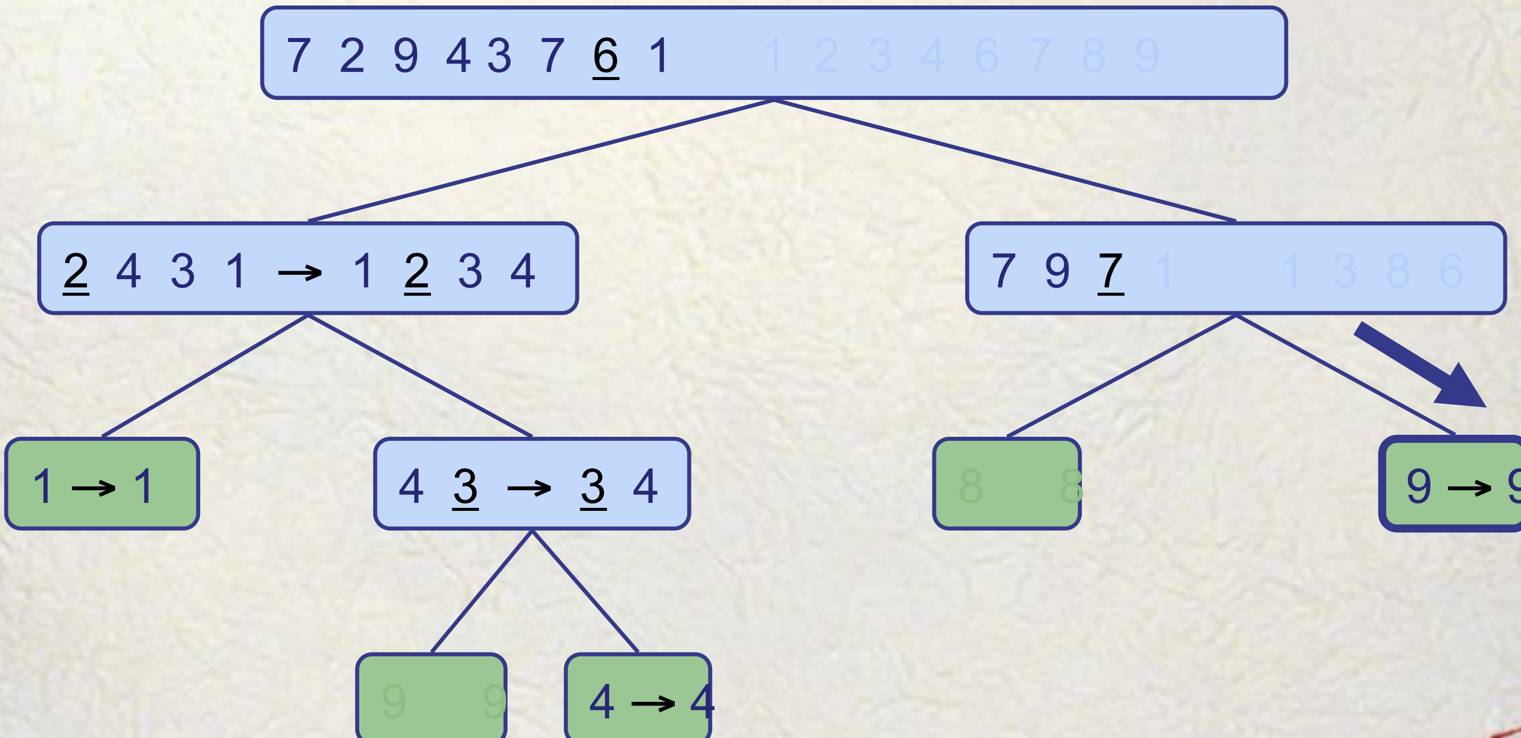
Execution Example (cont.)

- Recursive call, pivot selection



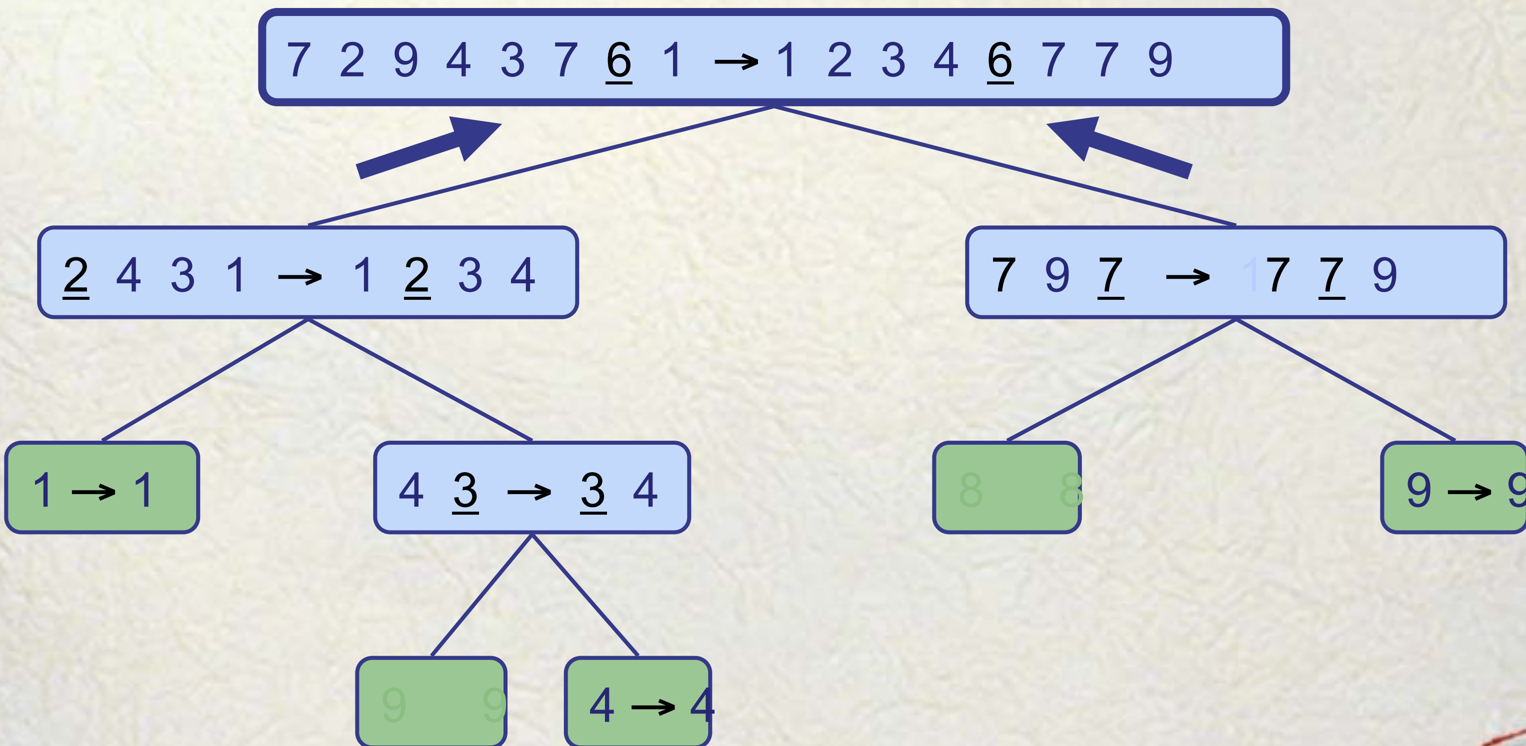
Execution Example (cont.)

- Partition, …, recursive call, base case



Execution Example (cont.)

- Join, join



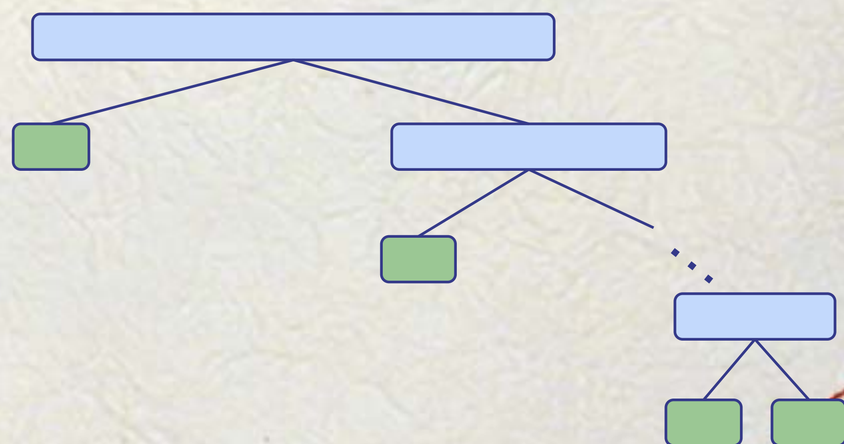
Worst-case Running Time

- The worst case for quick-sort occurs when the pivot is the unique minimum or maximum element
- One of L and G has size $n - 1$ and the other has size 0
- The running time is proportional to the sum

$$n + (n - 1) + \ldots + 2 + 1$$

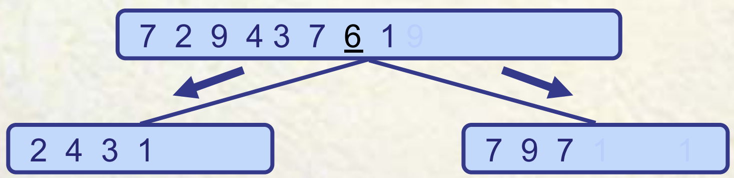
- Thus, the worst-case running time of quick-sort is $O(n^2)$

depth	time
0	n
1	$n - 1$
…	…
$n - 1$	1

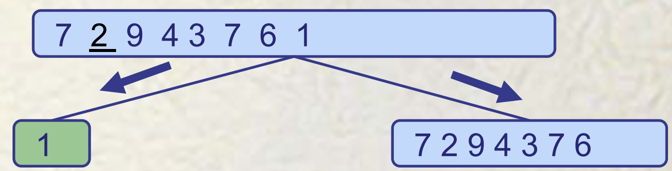


Expected Running Time

- Consider a recursive call of quick-sort on a sequence of size s
  - **Good call:** the sizes of L and G are each less than $3s/4$
  - **Bad call:** one of L and G has size greater than $3s/4$

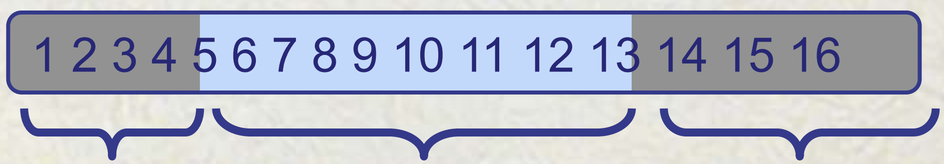


Good call



Bad call

- A call is good with probability $1/2$
  - 1/2 of the possible pivots cause good calls:

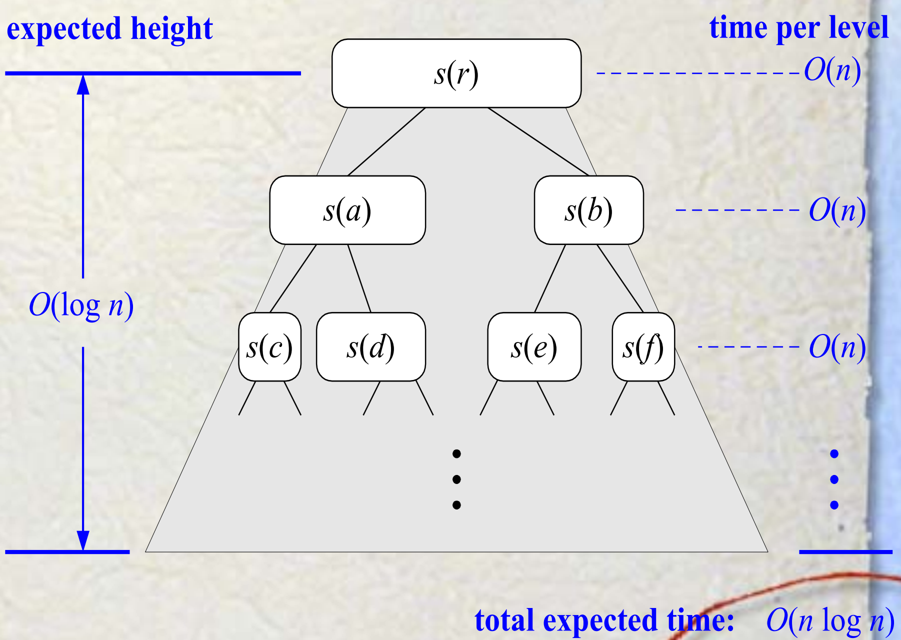


Bad pivots **Good pivots** **Bad pivots**

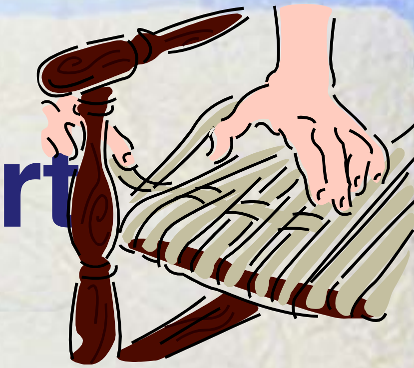
Expected Running Time, Part 2

- Probabilistic Fact: The expected number of coin tosses required in order to get k heads is $2k$
- For a node of with input size s, the input sizes of its children are each at most s3/4 or s/(4/3).

- Therefore, we have
  - For a node of depth $2\log_{4/3} n$, the expected input size is one
  - The expected height of the quick-sort tree is $O(\log n)$
- The amount or work done at the nodes of the same depth is $O(n)$
- Thus, the expected running time of quick-sort is $O(n \log n)$



In-Place Quick-Sort



- Quick-sort can be implemented to run in-place
- In the partition step, we use replace operations to rearrange the elements of the input sequence such that
  - the elements less than the pivot have rank less than h
  - the elements equal to the pivot have rank between h and k
  - the elements greater than the pivot have rank greater than k
- The recursive calls consider
  - elements with rank less than h
  - elements with rank greater than k

Algorithm *inPlaceQuickSort*(S, l, r)
    Input sequence S, ranks l and r
    Output sequence S with the elements of rank between l and r rearranged in increasing order
    if $l \geq r$
        return
    $i \leftarrow$ a random integer between l and r
    $x \leftarrow S.elemAtRank(i)$
    $(h, k) \leftarrow inPlacePartition(x)$
    $inPlaceQuickSort(S, l, h - 1)$
    $inPlaceQuickSort(S, k + 1, r)$

In-Place Partitioning

- Perform the partition using two indices to split S into L and E U G (a similar method can split E U G into E and G).

j k

3 2 5 1 0 7 3 5 9 2 7 9 8 9 7 **6** 9 (pivot = 6)

- Repeat until j and k cross:
  - Scan j to the right until finding an element $\geq$ x.
  - Scan k to the left until finding an element < x.
  - Swap elements at indices j and k

j k

3 2 5 1 0 $\boxed{7}$ 3 5 9 $\boxed{2}$ 7 9 8 9 7 **6** 9

Summary of Sorting Algorithms

Algorithm	Time	Notes
selection-sort	$O(n^2)$	◆ in-place ◆ slow (good for small inputs)
insertion-sort	$O(n^2)$	◆ in-place ◆ slow (good for small inputs)
quick-sort	$O(n \log n)$ expected	◆ in-place, randomized ◆ fastest (good for large inputs)
heap-sort	$O(n \log n)$	◆ in-place ◆ fast (good for large inputs)
merge-sort	$O(n \log n)$	◆ sequential data access ◆ fast (good for huge inputs)

Java Implementation

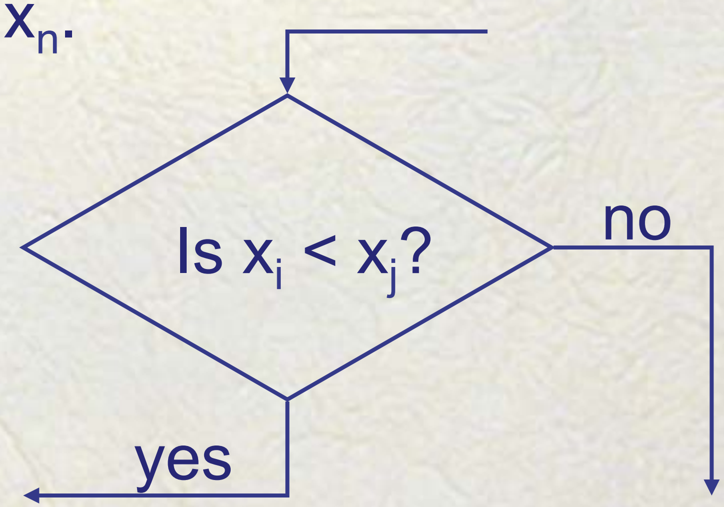
```
public static void quickSort (Object[] S, Comparator c) {
   if (S.length < 2) return; // the array is already sorted in this case
   quickSortStep(S, c, 0, S.length-1); // recursive sort method
 }
 private static void quickSortStep (Object[] S, Comparator c,
                          int leftBound, int rightBound ) {
   if (leftBound >= rightBound) return; // the indices have crossed
   Object temp;  // temp object used for swapping
   Object pivot = S[rightBound];
   int leftIndex = leftBound;     // will scan rightward
   int rightIndex = rightBound-1; // will scan leftward
   while (leftIndex <= rightIndex) { // scan right until larger than the pivot
     while ( (leftIndex <= rightIndex) && (c.compare(S[leftIndex], pivot)<=0) )
       leftIndex++;
     // scan leftward to find an element smaller than the pivot
     while ( (rightIndex >= leftIndex) && (c.compare(S[rightIndex], pivot)>=0))
       rightIndex--;
     if (leftIndex < rightIndex) { // both elements were found
       temp = S[rightIndex];
       S[rightIndex] = S[leftIndex]; // swap these elements
       S[leftIndex] = temp;
     }
   } // the loop continues until the indices cross
   temp = S[rightBound]; // swap pivot with the element at leftIndex
   S[rightBound] = S[leftIndex];
   S[leftIndex] = temp; // the pivot is now at leftIndex, so recurse
   quickSortStep(S, c, leftBound, leftIndex-1);
   quickSortStep(S, c, leftIndex+1, rightBound);
```

only works for distinct elements

Sorting Lower Bound

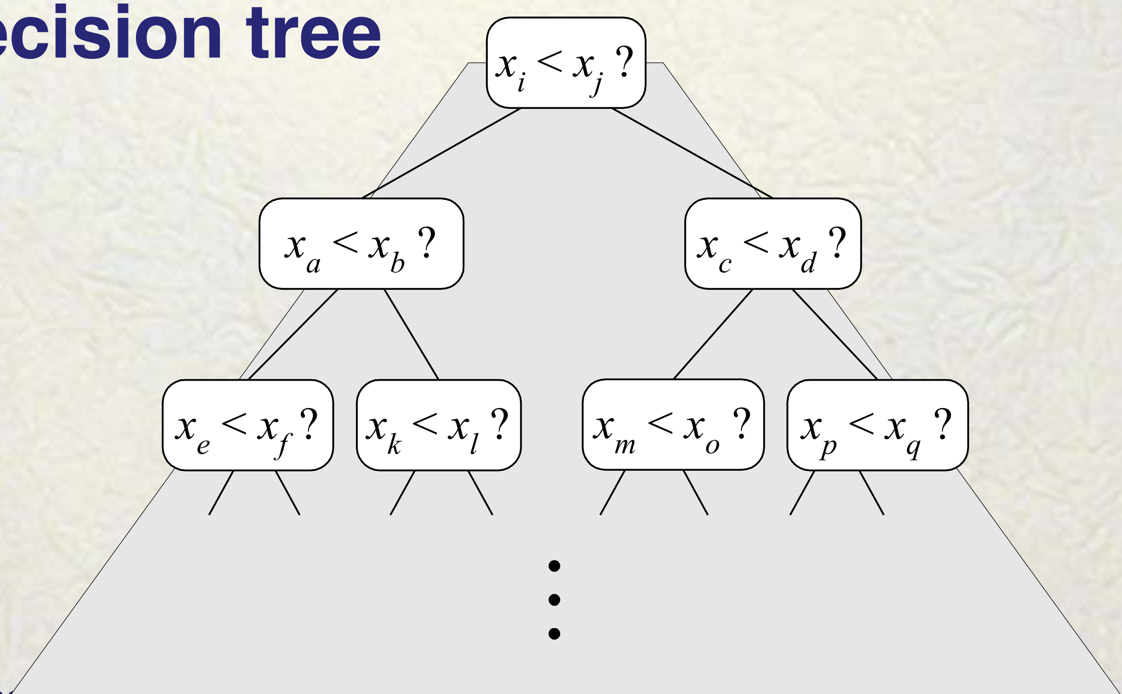
Comparison-Based Sorting

- Many sorting algorithms are comparison based.
  - They sort by making comparisons between pairs of objects
  - Examples: bubble-sort, selection-sort, insertion-sort, heap-sort, merge-sort, quick-sort, ...
- Let us therefore derive a lower bound on the running time of any algorithm that uses comparisons to sort n elements, $x_1, x_2, \ldots, x_n$.



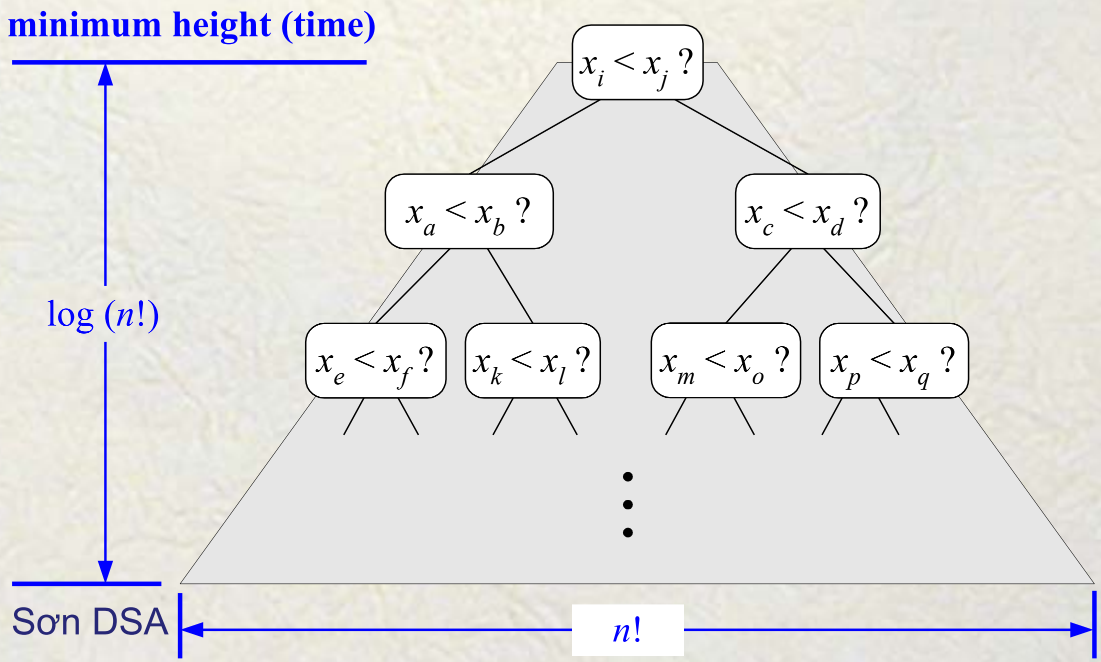
Counting Comparisons

- Let us just count comparisons then.
- Each possible run of the algorithm corresponds to a root-to-leaf path in a **decision tree**

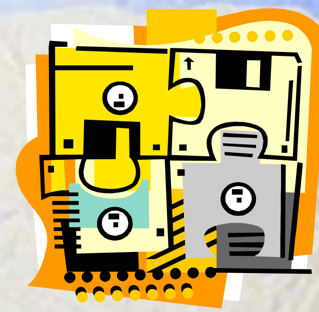


Decision Tree Height

- The height of this decision tree is a lower bound on the running time
- Every possible input permutation must lead to a separate leaf output.
  – If not, some input …4…5… would have same output ordering as …5…4…, which would be wrong.
- Since there are n!=1*2*…*n leaves, the height is at least log (n!)



The Lower Bound

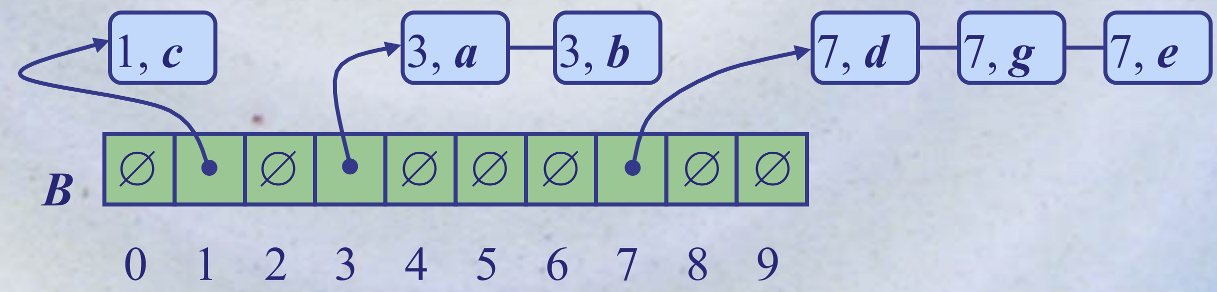


- Any comparison-based sorting algorithms takes at least log (n!) time
- Therefore, any such algorithm takes time at least

$$\log\,(n!) \geq \log\left(\frac{n}{2}\right)^{\frac{n}{2}} = (n/2)\log(n/2).$$

- That is, any comparison-based sorting algorithm must run in Ω(n log n) time.

Bucket-Sort and Radix-Sort



Bucket-Sort

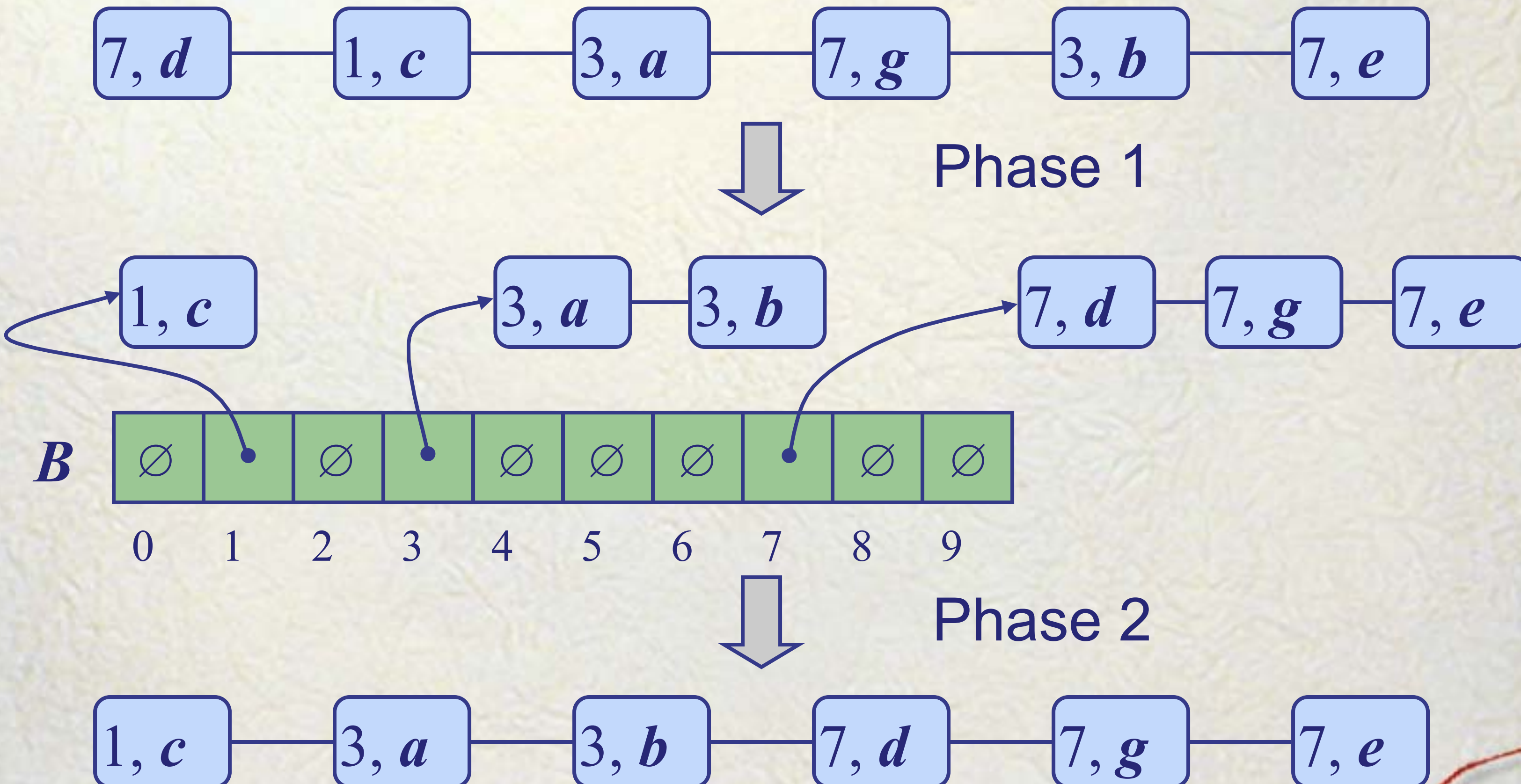
- Let be S be a sequence of n (key, element) entries with keys in the range $[0, N-1]$
- Bucket-sort uses the keys as indices into an auxiliary array B of sequences (buckets)
  - Phase 1: Empty sequence S by moving each entry (k, o) into its bucket $B[k]$
  - Phase 2: For $i = 0, \ldots, N-1$, move the entries of bucket $B[i]$ to the end of sequence S
- Analysis:
  - Phase 1 takes $O(n)$ time
  - Phase 2 takes $O(n + N)$ time

Bucket-sort takes $O(n + N)$ time

Algorithm *bucketSort*(S, N)
  Input sequence S of (key, element) items with keys in the range $[0, N-1]$
  Output sequence S sorted by increasing keys
  $B \leftarrow$ array of N empty sequences
  while $\neg S.isEmpty()$
    $f \leftarrow S.first()$
    $(k, o) \leftarrow S.remove(f)$
    $B[k].insertLast((k, o))$
  for $i \leftarrow 0$ **to** $N-1$
    while $\neg B[i].isEmpty()$
      $f \leftarrow B[i].first()$
      $(k, o) \leftarrow B[i].remove(f)$
      $S.insertLast((k, o))$

Example

- Key range $[0, 9]$



Properties and Extensions

- Key-type Property
  - The keys are used as indices into an array and cannot be arbitrary objects
  - No external comparator
- **Stable** Sort Property
  - The relative order of any two items with the same key is preserved after the execution of the algorithm

Extensions
- Integer keys in the range $[a, b]$
  - Put entry (k, o) into bucket $B[k - a]$
- String keys from a set D of possible strings, where D has constant size (e.g., names of the 50 U.S. states)
  - Sort D and compute the rank $r(k)$ of each string k of D in the sorted sequence
  - Put entry (k, o) into bucket $B[r(k)]$

Lexicographic Order

- A d-tuple is a sequence of d keys $(k_1, k_2, \ldots, k_d)$, where key k_i is said to be the i-th dimension of the tuple
- Example:
  - The Cartesian coordinates of a point in space are a 3-tuple
- The lexicographic order of two d-tuples is recursively defined as follows

$$(x_1, x_2, \ldots, x_d) < (y_1, y_2, \ldots, y_d)$$

$$\Leftrightarrow$$

$$x_1 < y_1 \lor x_1 = y_1 \land (x_2, \ldots, x_d) < (y_2, \ldots, y_d)$$

I.e., the tuples are compared by the first dimension, then by the second dimension, etc.

Lexicographic-Sort

- Let C_i be the comparator that compares two tuples by their ***i***-th dimension
- Let ***stableSort***(***S***, ***C***) be a stable sorting algorithm that uses comparator ***C***
- Lexicographic-sort sorts a sequence of ***d***-tuples in lexicographic order by executing ***d*** times algorithm ***stableSort***, one per dimension
- Lexicographic-sort runs in $O(dT(n))$ time, where $T(n)$ is the running time of ***stableSort***

Algorithm ***lexicographicSort***(***S***)
    Input sequence ***S*** of ***d***-tuples
    Output sequence ***S*** sorted in lexicographic order

    for $i \leftarrow d$ **downto** 1
        stableSort(***S***, C_i)

Example:

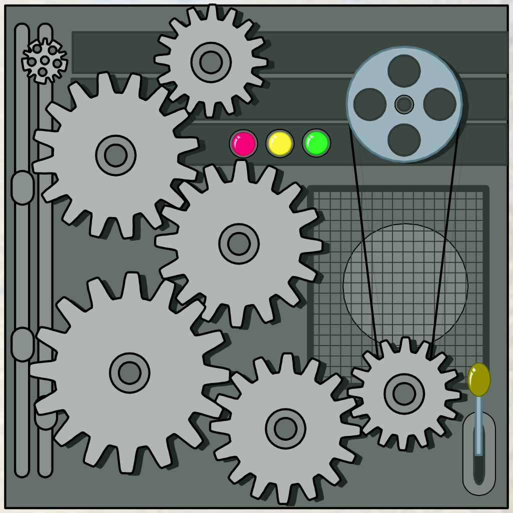
(7,4,6) (5,1,5) (2,4,6) (2, 1, 4) (3, 2, 4)

(2, 1, 4) (3, 2, 4) (5,1,5) (7,4,6) (2,4,6)

(2, 1, 4) (5,1,5) (3, 2, 4) (7,4,6) (2,4,6)

(2, 1, 4) (2,4,6) (3, 2, 4) (5,1,5) (7,4,6)

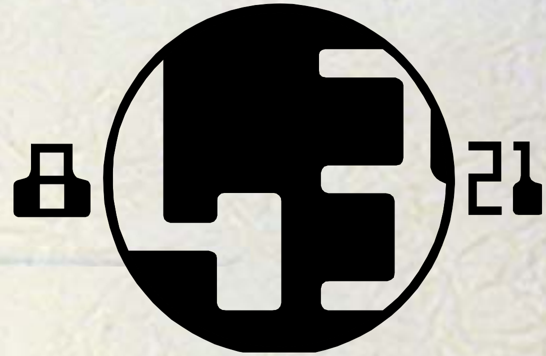
Radix-Sort



- Radix-sort is a specialization of lexicographic-sort that uses bucket-sort as the stable sorting algorithm in each dimension
- Radix-sort is applicable to tuples where the keys in each dimension i are integers in the range $[0, N-1]$
- Radix-sort runs in time $O(d(n+N))$

Algorithm *radixSort*(S, N)
  Input sequence S of d-tuples such that $(0, \ldots, 0) \le (x_1, \ldots, x_d)$ and $(x_1, \ldots, x_d) \le (N-1, \ldots, N-1)$ for each tuple $(x_1, \ldots, x_d)$ in S
  Output sequence S sorted in lexicographic order
  for $i \leftarrow d$ **downto** 1
    bucketSort(S, N)

Radix-Sort for Binary Numbers



- Consider a sequence of n b-bit integers
  $$x = x_{b-1} \dots x_1 x_0$$
- We represent each element as a b-tuple of integers in the range $[0, 1]$ and apply radix-sort with $N = 2$
- This application of the radix-sort algorithm runs in $O(bn)$ time
- For example, we can sort a sequence of 32-bit integers in linear time

Algorithm ***binaryRadixSort***(S)
  Input sequence S of b-bit integers
  Output sequence S sorted
  replace each element x of S with the item $(0, x)$
  for $i \leftarrow 0$ **to** $b - 1$
    replace the key k of each item (k, x) of S with bit x_i of x
  bucketSort$(S, 2)$

Example

- Sorting a sequence of 4-bit integers