Artificial Intelligence

Inference in First-order Logic

Outline

- Reducing first-order inference to propositional inference
- Unification
- Generalized Modus Ponens
- Forward chaining
- Backward chaining
- Resolution

Universal instantiation (UI)

- Every instantiation of a universally quantified sentence is entailed by it:

$$\frac{\forall v\ \alpha}{\text{Subst}(\{v/g\}, \alpha)}$$

  for any variable *v* and ground term *g*
- E.g., ∀x *King*(*x*) ∧ *Greedy*(*x*) ⇒ *Evil*(*x*) yields:

  King(*John*) ∧ *Greedy*(*John*) ⇒ *Evil*(*John*)

  King(*Richard*) ∧ *Greedy*(*Richard*) ⇒ *Evil*(*Richard*)

  King(*Father*(*John*)) ∧ *Greedy*(*Father*(*John*)) ⇒ *Evil*(*Father*(*John*))

  .

  .

  .

Existential instantiation (EI)

- For any sentence α, variable *v*, and constant symbol *k* that does not appear elsewhere in the knowledge base:

$$\frac{\exists v\ \alpha}{\text{Subst}(\{v/k\}, \alpha)}$$

- E.g., $\exists x\ Crown(x) \wedge OnHead(x, John)$ yields:

$$Crown(C_1) \wedge OnHead(C_1, John)$$

provided C_1 is a new constant symbol, called a Skolem constant

Reduction to propositional inference

Suppose the KB contains just the following:

$\forall x$ King(x) $\wedge$ Greedy(x) $\Rightarrow$ Evil(x)
King(John)
Greedy(John)
Brother(Richard,John)

- Instantiating the universal sentence in all possible ways, we have:

  King(John) $\wedge$ Greedy(John) $\Rightarrow$ Evil(John)
  King(Richard) $\wedge$ Greedy(Richard) $\Rightarrow$ Evil(Richard)
  King(John)
  Greedy(John)
  Brother(Richard,John)

- The new KB is propositionalized: proposition symbols are

  King(John), Greedy(John), Evil(John), King(Richard), etc.

Reduction contd.

- Every FOL KB can be propositionalized so as to preserve entailment
- (A ground sentence is entailed by new KB iff entailed by original KB)
- Idea: propositionalize KB and query, apply resolution, return result
- Problem: with function symbols, there are infinitely many ground terms,
  - e.g., *Father*(*Father*(*Father*(*John*)))

Reduction contd.

Theorem: Herbrand (1930). If a sentence α is entailed by an FOL KB, it is entailed by a finite subset of the propositionalized KB

Idea: For $n = 0$ to ∞ do

create a propositional KB by instantiating with depth-n terms

see if α is entailed by this KB

Problem: works if α is entailed, loops if α is not entailed

Theorem: Turing (1936), Church (1936) Entailment for FOL is semidecidable (algorithms exist that say yes to every entailed sentence, but no algorithm exists that also says no to every nonentailed sentence.)

Problems with propositionalization

- Propositionalization seems to generate lots of irrelevant sentences.
- E.g., from:

  $\forall x\ King(x) \wedge Greedy(x) \Rightarrow Evil(x)$
  King(John)
  $\forall y\ Greedy(y)$
  Brother(Richard,John)

- it seems obvious that *Evil*(*John*), but propositionalization produces lots of facts such as *Greedy*(*Richard*) that are irrelevant

- With *p* *k*-ary predicates and *n* constants, there are $p \cdot n^k$ instantiations.

Unification

- We can get the inference immediately if we can find a substitution θ such that *King(x)* and *Greedy(x)* match *King(John)* and *Greedy(y)*

θ = {x/John,y/John} works

- Unify(α,β) = θ if αθ = βθ

p	q	θ
Knows(John,x)	Knows(John,Jane)	
Knows(John,x)	Knows(y,OJ)	
Knows(John,x)	Knows(y,Mother(y))	
Knows(John,x)	Knows(x,OJ)	

- Standardizing apart eliminates overlap of variables, e.g., Knows(z_{17},OJ)

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p	q	θ
Knows(John,x)	Knows(John,Jane)	{x/Jane}}
Knows(John,x)	Knows(y,OJ)	{x/OJ,y/John}}
Knows(John,x)	Knows(y,Mother(y))	{y/John,x/Mother(John)}}
Knows(John,x)	Knows(x,OJ)	{fail}

- Standardizing apart eliminates overlap of variables, e.g., Knows(z_{17},OJ)

Unification

- To unify *Knows(John,x)* and *Knows(y,z)*,
  θ = {y/John, x/z } or θ = {y/John, x/John, z/John}
- The first unifier is more general than the second.
- There is a single most general unifier (MGU) that is unique up to renaming of variables.
  MGU = { y/John, x/z }

The unification algorithm

```
function UNIFY(x, y, θ) returns a substitution to make x and y identical
   inputs: x, a variable, constant, list, or compound
           y, a variable, constant, list, or compound
           θ, the substitution built up so far

   if θ = failure then return failure
   else if x = y then return θ
   else if VARIABLE?(x) then return UNIFY-VAR(x, y, θ)
   else if VARIABLE?(y) then return UNIFY-VAR(y, x, θ)
   else if COMPOUND?(x) and COMPOUND?(y) then
       return UNIFY(ARGS[x], ARGS[y], UNIFY(OP[x], OP[y], θ))
   else if LIST?(x) and LIST?(y) then
       return UNIFY(REST[x], REST[y], UNIFY(FIRST[x], FIRST[y], θ))
   else return failure
```

The unification algorithm

function UNIFY-VAR(var, x, θ) **returns** a substitution
  inputs: var, a variable
    x, any expression
    θ, the substitution built up so far

  if $\{var/val\} \in \theta$ **then return** UNIFY(val, x, θ)
  else if $\{x/val\} \in \theta$ **then return** UNIFY(var, val, θ)
  else if OCCUR-CHECK?(var, x) **then return** failure
  else return add $\{var/x\}$ to θ

Generalized Modus Ponens (GMP)

$$\frac{p_1', p_2', \ldots, p_n', (p_1 \wedge p_2 \wedge \ldots \wedge p_n \Rightarrow q)}{q\theta} \quad \text{where } p_i'\theta = p_i\theta \text{ for all } i$$

p_1' is *King*(*John*) p_1 is *King*(*x*)

p_2' is *Greedy*(*y*) p_2 is *Greedy*(*x*)

θ is {x/John,y/John} q is *Evil*(*x*)

q θ is *Evil*(*John*)

- GMP used with KB of definite clauses (exactly one positive literal)
- All variables assumed universally quantified

Soundness of GMP

- Need to show that

$$p_1', \ldots, p_n', (p_1 \land \ldots \land p_n \Rightarrow q) \models q\theta$$

provided that $p_i'\theta = p_i\theta$ for all *i*

- Lemma: For any sentence *p*, we have $p \models p\theta$ by UI

1. $(p_1 \land \ldots \land p_n \Rightarrow q) \models (p_1 \land \ldots \land p_n \Rightarrow q)\theta = (p_1\theta \land \ldots \land p_n\theta \Rightarrow q\theta)$
2. $p_1', \ldots, p_n' \models p_1' \land \ldots \land p_n' \models p_1'\theta \land \ldots \land p_n'\theta$
3. From 1 and 2, $q\theta$ follows by ordinary Modus Ponens

Example knowledge base

- The law says that it is a crime for an American to sell weapons to hostile nations. The country Nono, an enemy of America, has some missiles, and all of its missiles were sold to it by Colonel West, who is American.

- Prove that Col. West is a criminal

Example knowledge base contd.

... it is a crime for an American to sell weapons to hostile nations:

American(x) $\wedge$ *Weapon(y)* $\wedge$ *Sells(x,y,z)* $\wedge$ *Hostile(z)* $\Rightarrow$ *Criminal(x)*

Nono … has some missiles, i.e., $\exists x$ Owns(Nono,x) $\wedge$ Missile(x):

$Owns(Nono,M_1) \wedge Missile(M_1)$

… all of its missiles were sold to it by Colonel West

Missile(x) $\wedge$ *Owns(Nono,x)* $\Rightarrow$ *Sells(West,x,Nono)*

Missiles are weapons:

Missile(x) $\Rightarrow$ *Weapon(x)*

An enemy of America counts as "hostile“:

Enemy(x,America) $\Rightarrow$ *Hostile(x)*

West, who is American …

American(West)

The country Nono, an enemy of America …

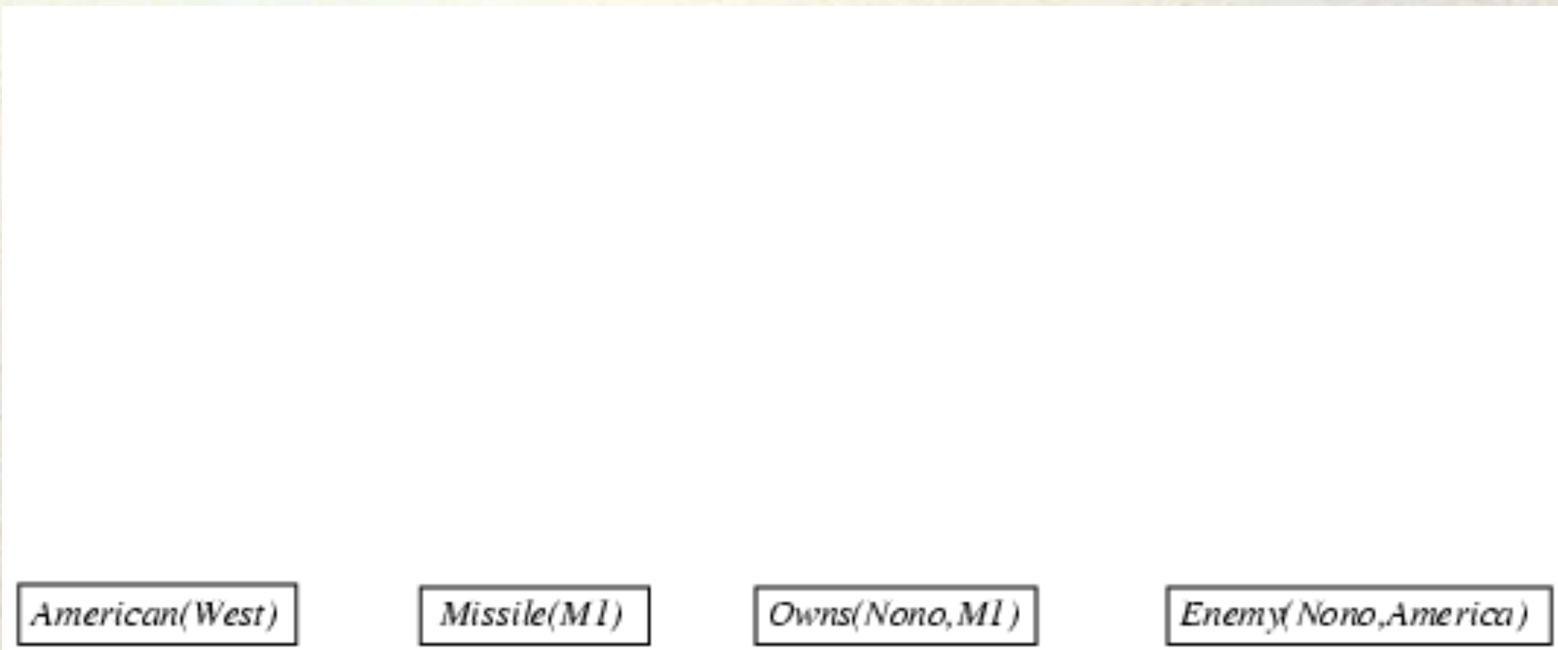
Enemy(Nono,America)

Forward chaining algorithm

function FOL-FC-ASK(KB, α) **returns** a substitution or *false*

  repeat until *new* is empty
    $new \leftarrow \{\}$
    for each sentence r **in** KB **do**
      $(p_1 \wedge \ldots \wedge p_n \Rightarrow q) \leftarrow$ STANDARDIZE-APART(r)
      for each θ such that $(p_1 \wedge \ldots \wedge p_n)\theta = (p'_1 \wedge \ldots \wedge p'_n)\theta$
            for some $p'_1, \ldots, p'_n$ in KB
        $q' \leftarrow$ SUBST(θ, q)
        if q' is not a renaming of a sentence already in KB or *new* **then do**
          add q' to *new*
          $\phi \leftarrow$ UNIFY(q', α)
          if ϕ is not *fail* **then return** ϕ
    add *new* to KB
  return *false*

Forward chaining proof



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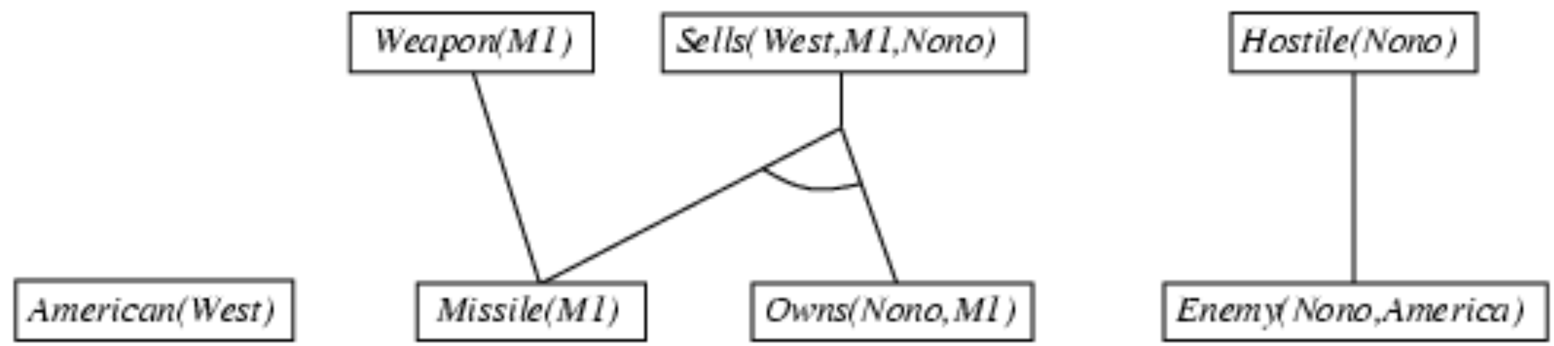

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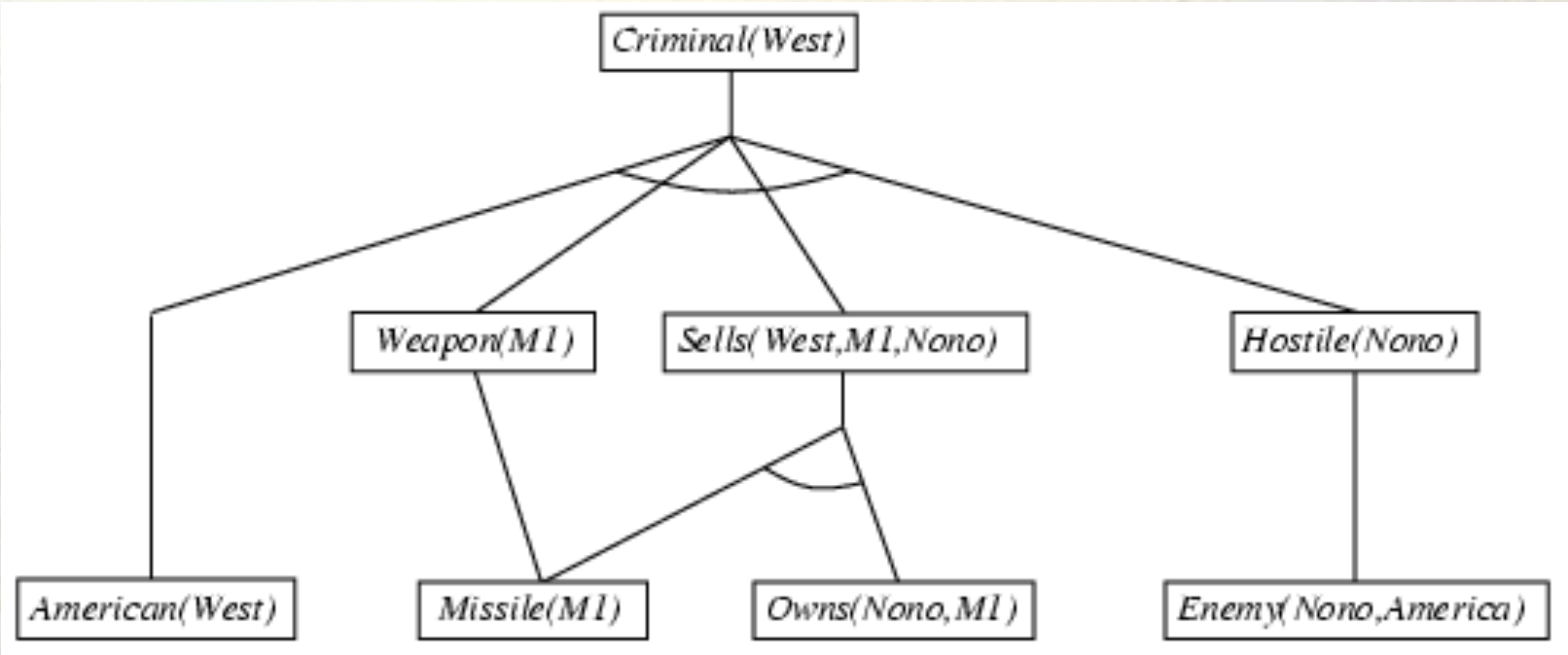


Properties of forward chaining

- Sound and complete for first-order definite clauses
- Datalog = first-order definite clauses + no functions
- FC terminates for Datalog in finite number of iterations
- May not terminate in general if α is not entailed
- This is unavoidable: entailment with definite clauses is semidecidable

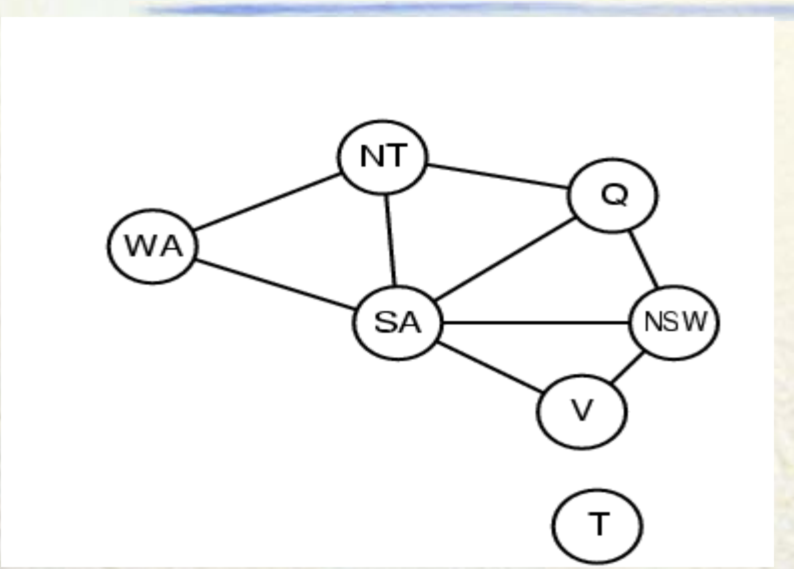
Efficiency of forward chaining

- Outer loop – recheck every rule:
  - Incremental forward chaining: no need to match a rule on iteration *k* if a premise wasn't added on iteration *k-1*
  - ⇒ match each rule whose premise contains a newly added positive literal
- Inner loop: Matching itself can be expensive:
  - Database indexing allows O(1) retrieval of known facts

    e.g., query *Missile(x)* retrieves $Missile(M_1)$
  - *Order of matching is important like in CSP – should we match Missile or Owns first:*

    Missile(x) ∧ Owns(Nono, x) -> Sell(West, x, Nono)
- Generate irrelevant facts:
  - Backward chaining
- Forward chaining is widely used in deductive databases

Hard matching example



Diff(wa,nt) ∧ *Diff(wa,sa)* ∧ *Diff(nt,q)* ∧ *Diff(nt,sa)* ∧ *Diff(q,nsw)* ∧ *Diff(q,sa)* ∧ *Diff(nsw,v)* ∧ *Diff(nsw,sa)* ∧ *Diff(v,sa)* ⇒ *Colorable()*

Diff(Red,Blue) *Diff (Red,Green)*
Diff(Green,Red) *Diff(Green,Blue)*
Diff(Blue,Red) *Diff(Blue,Green)*

- *Colorable*() is inferred iff the CSP has a solution
- CSPs include 3SAT as a special case, hence matching is NP-hard

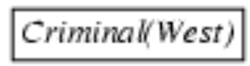
Backward chaining algorithm

function FOL-BC-ASK(KB, goals, θ) **returns** a set of substitutions
  inputs: KB, a knowledge base
    $goals$, a list of conjuncts forming a query
    θ, the current substitution, initially the empty substitution $\{\ \}$
  local variables: ans, a set of substitutions, initially empty

  if $goals$ is empty **then return** $\{\theta\}$
  $q' \leftarrow \text{SUBST}(\theta, \text{FIRST}(goals))$
  for each r **in** KB where STANDARDIZE-APART(r) = $(p_1 \land \ldots \land p_n \Rightarrow q)$
    and $\theta' \leftarrow \text{UNIFY}(q, q')$ succeeds
   $ans \leftarrow \text{FOL-BC-ASK}(KB, [p_1, \ldots, p_n | \text{REST}(goals)], \text{COMPOSE}(\theta, \theta')) \cup ans$
  return ans

$\text{SUBST}(\text{COMPOSE}(\theta_1, \theta_2), p) = \text{SUBST}(\theta_2, \text{SUBST}(\theta_1, p))$

Backward chaining example



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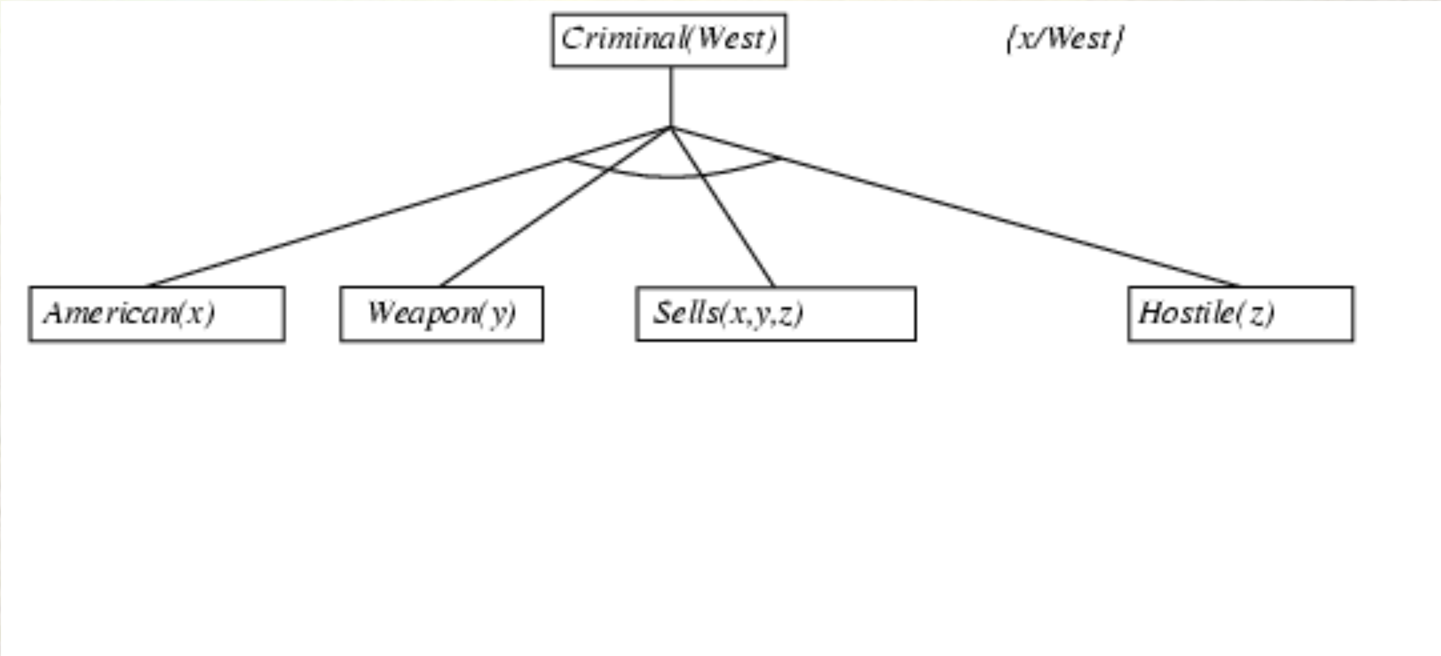

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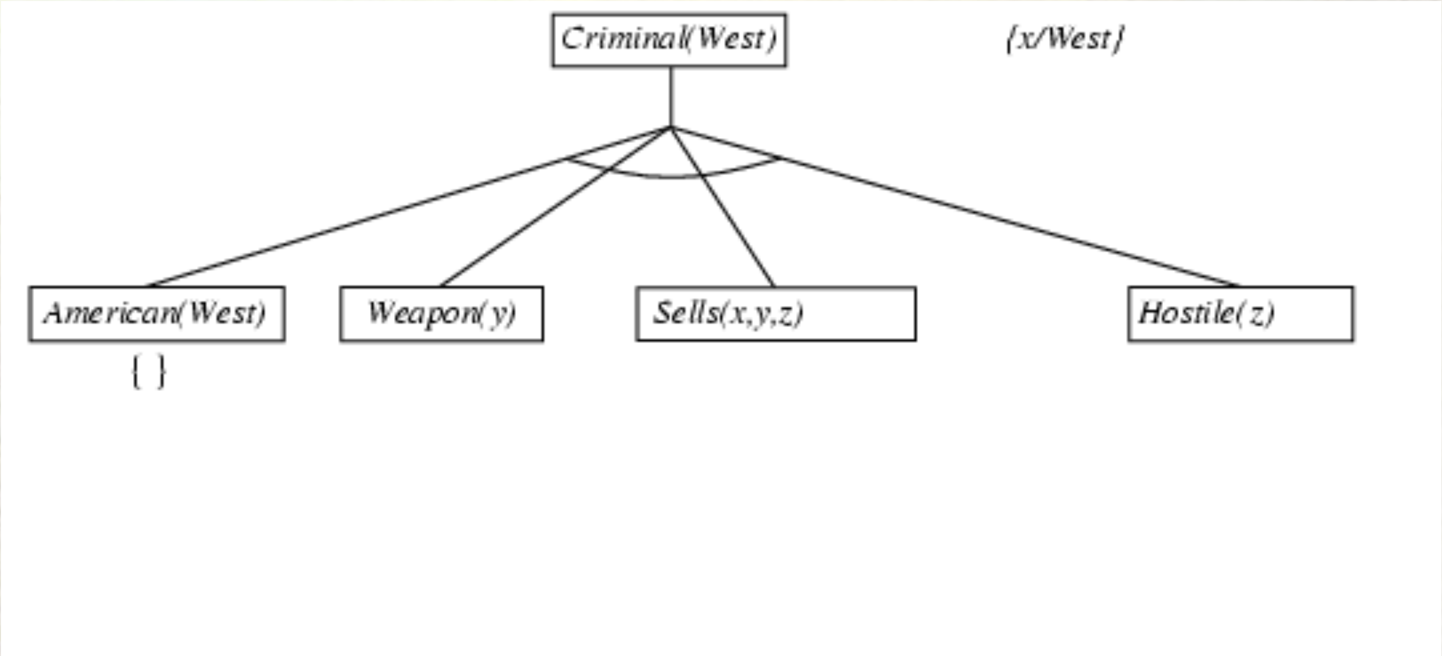

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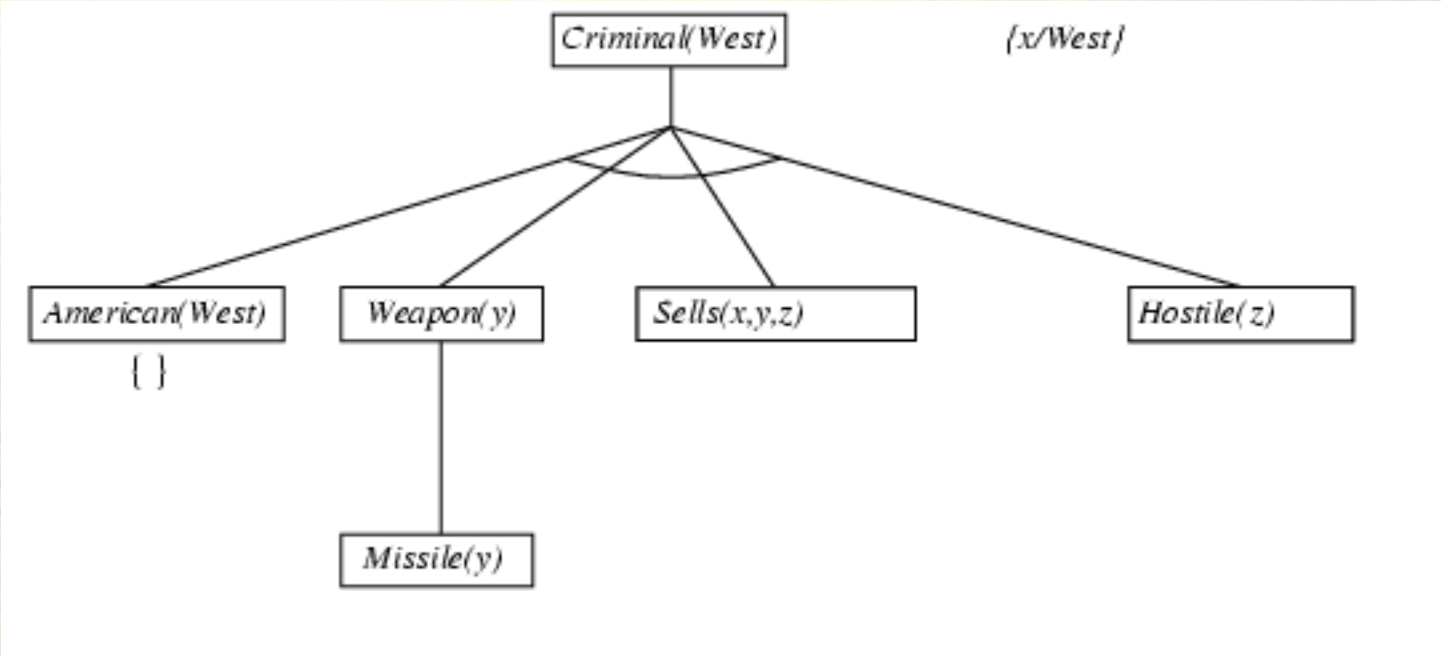

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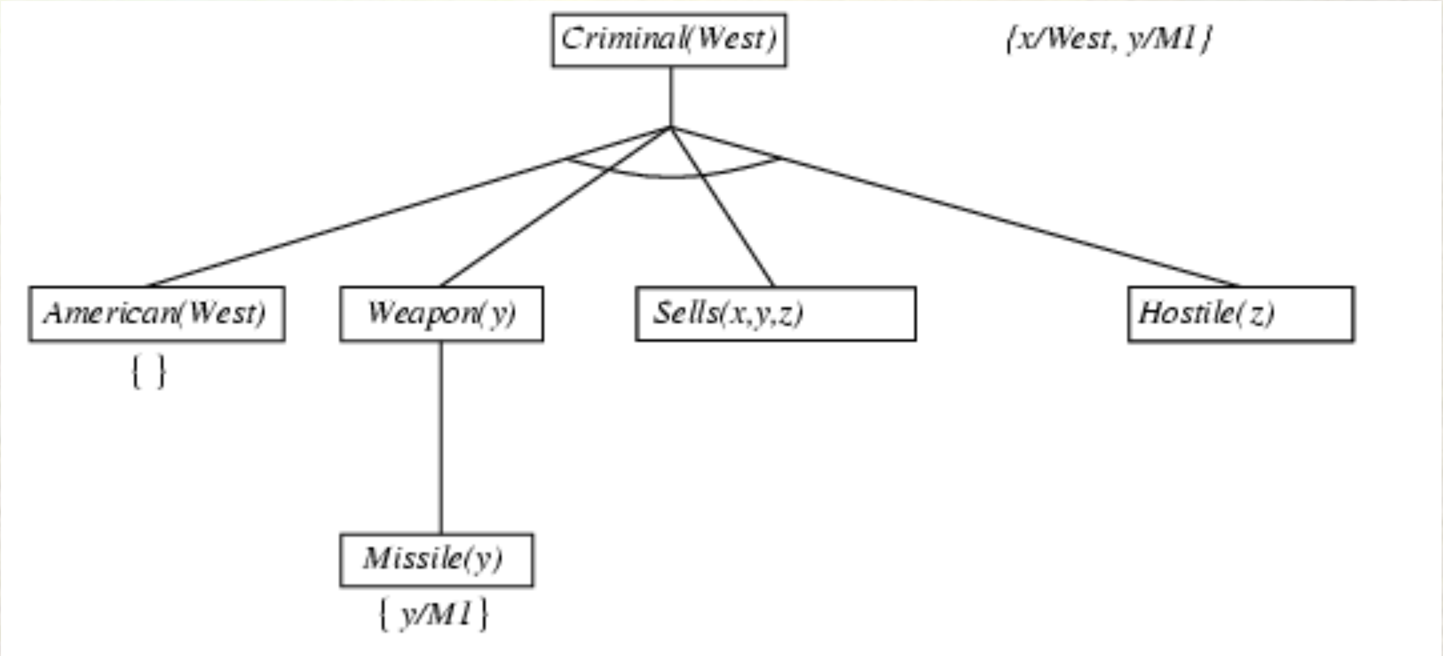

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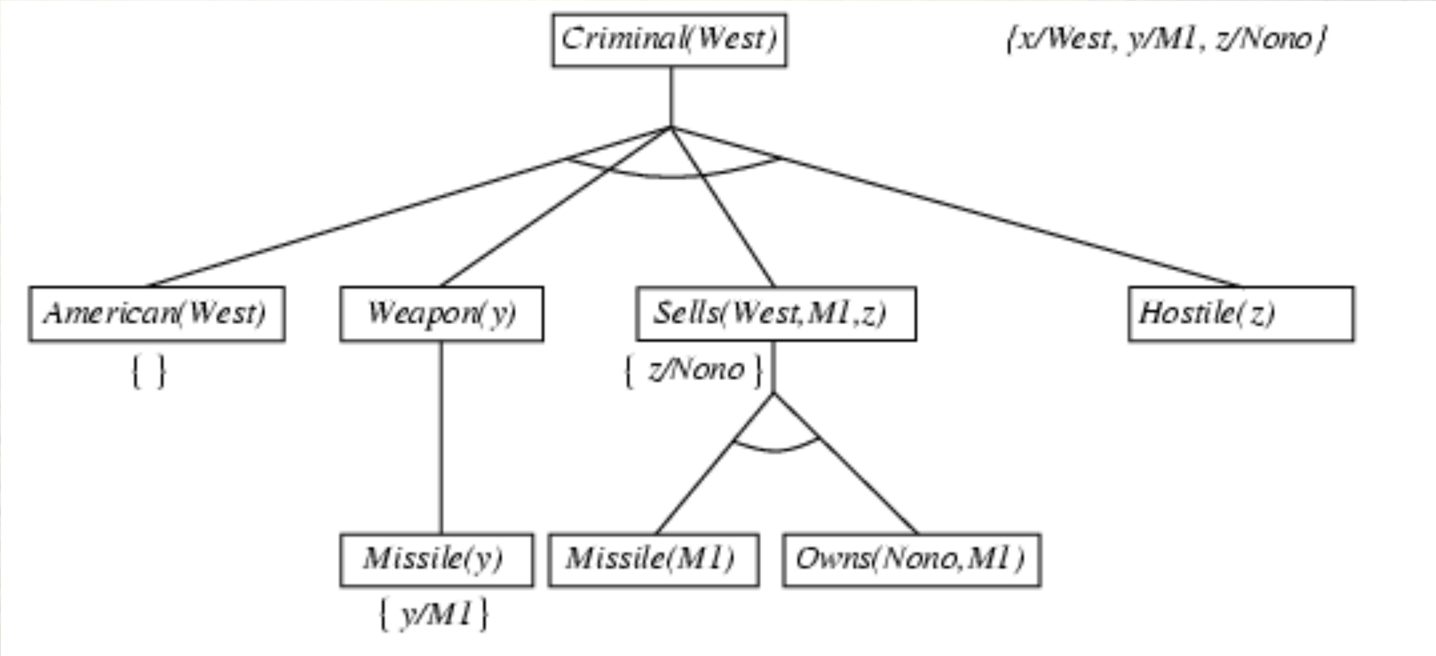

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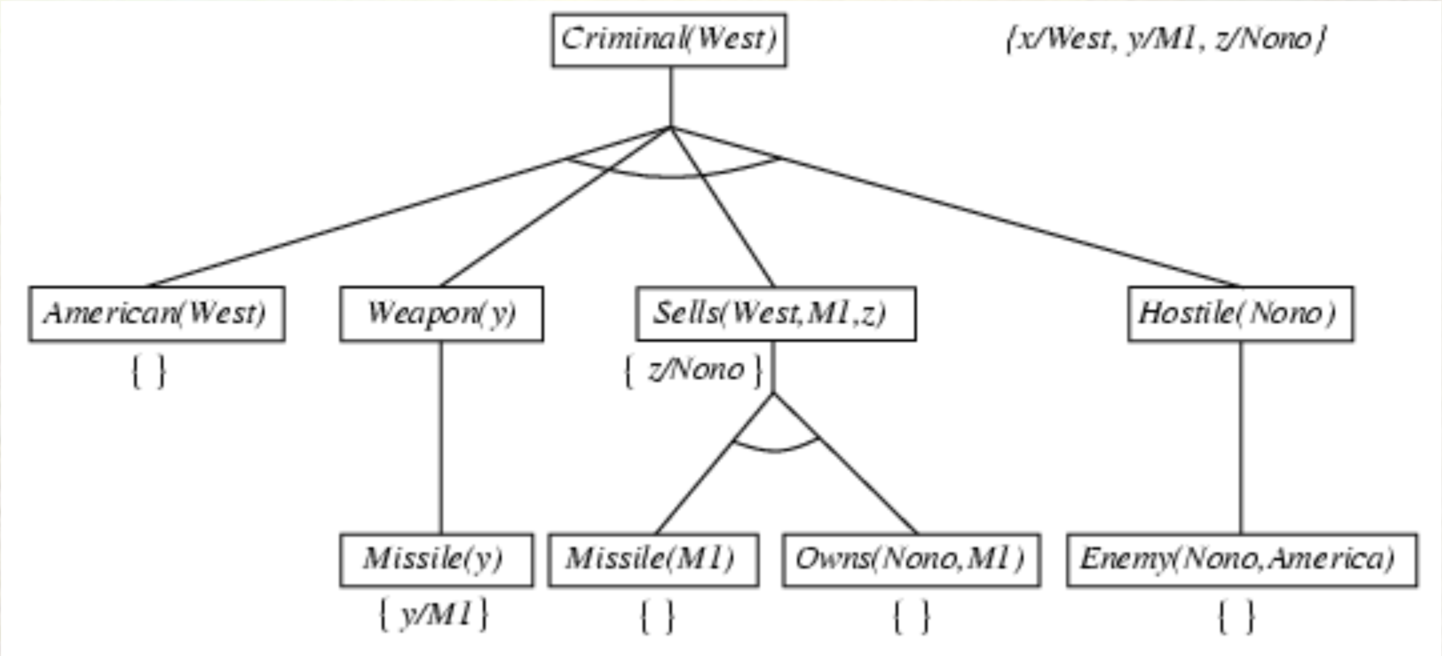

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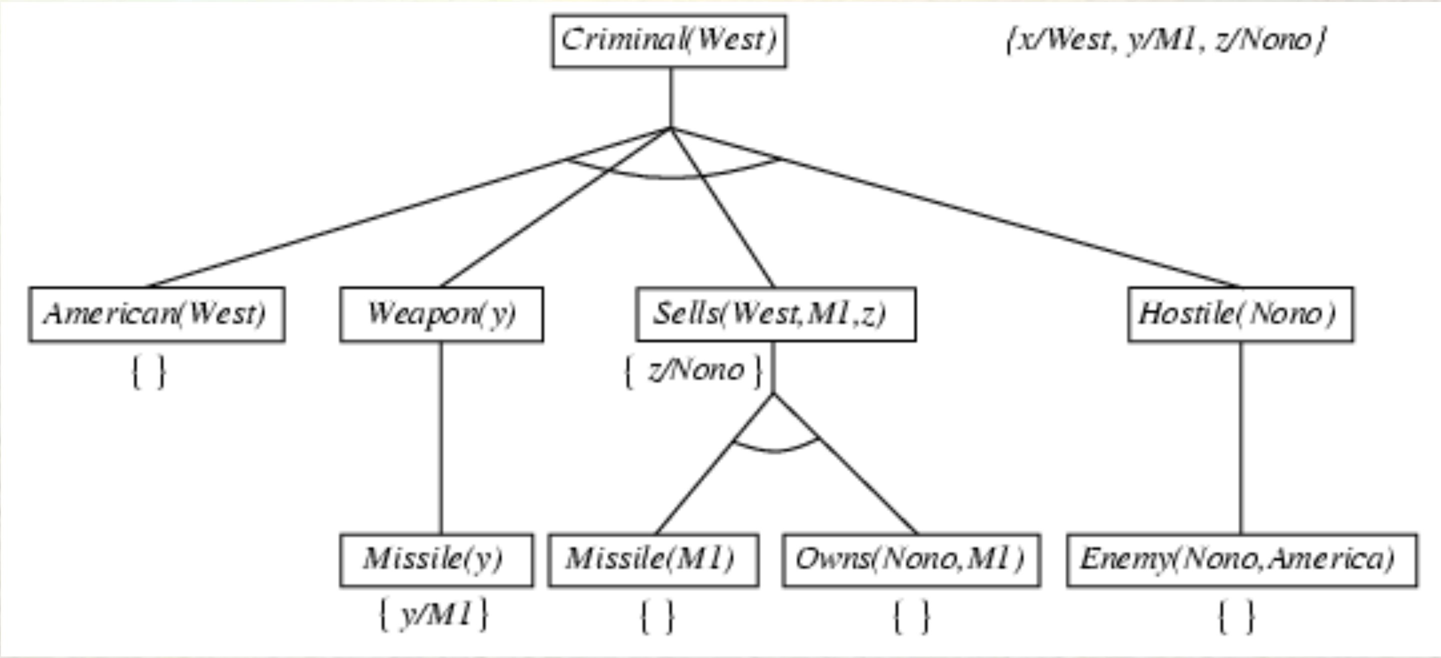

Properties of backward chaining

- Depth-first recursive proof search: space is linear in size of proof
- Incomplete due to infinite loops
  - $\Rightarrow$ fix by checking current goal against every goal on stack
- Inefficient due to repeated subgoals (both success and failure)
  - $\Rightarrow$ fix using caching of previous results (extra space)
- Widely used for logic programming

Resolution: brief summary

- Full first-order version:
- $$\frac{\ell_1 \vee \cdots \vee \ell_k, \qquad m_1 \vee \cdots \vee m_n}{(\ell_1 \vee \cdots \vee \ell_{i-1} \vee \ell_{i+1} \vee \cdots \vee \ell_k \vee m_1 \vee \cdots \vee m_{j-1} \vee m_{j+1} \vee \cdots \vee m_n)\theta}$$
  where Unify(ℓ_i, $\neg m_j$) = θ.

- The two clauses are assumed to be standardized apart so that they share no variables.
- For example,

$$\frac{\neg Rich(x) \vee Unhappy(x) \qquad Rich(Ken)}{Unhappy(Ken)}$$

  with θ = {x/Ken}

- Apply resolution steps to CNF(KB $\wedge$ $\neg\alpha$); complete for FOL

Conversion to CNF

- Everyone who loves all animals is loved by someone:

  $\forall x\ [\forall y\ Animal(y) \Rightarrow Loves(x,y)] \Rightarrow [\exists y\ Loves(y,x)]$

- 1. Eliminate biconditionals and implications

  $\forall x\ \neg [\forall y\ \neg Animal(y) \lor Loves(x,y)] \lor [\exists y\ Loves(y,x)]$

- 2. Move $\neg$ inwards: $\neg \forall x\ p \equiv \exists x\ \neg p$, $\neg \exists x\ p \equiv \forall x\ \neg p$

  $\forall x\ [\exists y\ \neg(\neg Animal(y) \lor Loves(x,y))] \lor [\exists y\ Loves(y,x)]$

  $\forall x\ [\exists y\ \neg\neg Animal(y) \land \neg Loves(x,y)] \lor [\exists y\ Loves(y,x)]$

  $\forall x\ [\exists y\ Animal(y) \land \neg Loves(x,y)] \lor [\exists y\ Loves(y,x)]$

Conversion to CNF contd.

3. Standardize variables: each quantifier should use a different one

   $\forall x\ [\exists y\ Animal(y) \wedge \neg Loves(x,y)] \vee [\exists z\ Loves(z,x)]$

4. Skolemize: a more general form of existential instantiation.

   Each existential variable is replaced by a Skolem function of the enclosing universally quantified variables:

   $\forall x\ [Animal(F(x)) \wedge \neg Loves(x,F(x))] \vee Loves(G(x),x)$

7. Drop universal quantifiers:

   $[Animal(F(x)) \wedge \neg Loves(x,F(x))] \vee Loves(G(x),x)$

9. Distribute $\vee$ over $\wedge$:

   $[Animal(F(x)) \vee Loves(G(x),x)] \wedge [\neg Loves(x,F(x)) \vee Loves(G(x),x)]$

Resolution proof: definite clauses