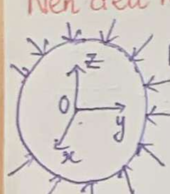
PTTTBD Saint - Venant

$$\frac{\partial^2 \varepsilon_x}{\partial y^2}+\frac{\partial^2 \varepsilon_y}{\partial x^2}=2\cdot\frac{\partial^2 \varepsilon_{xy}}{\partial x\partial y}\ \ldots$$

$$\frac{\partial^2 \varepsilon_x}{\partial y\partial z}=\frac{\partial}{\partial x}\left(-\frac{\partial \varepsilon_{yz}}{\partial x}+\frac{\partial \varepsilon_{zx}}{\partial y}+\frac{\partial \varepsilon_{xy}}{\partial z}\right)\ldots$$

Nén đều mọi phía



p - ĐKB: $\sigma_x=\sigma_y=\sigma_z=-p$

$\tau_{xy}=\tau_{yz}=\tau_{zx}=0$

- Hooke ngược.

$\varepsilon_x=\varepsilon_y=\varepsilon_z=\frac{-p(1-2\nu)}{E}=A$

$\varepsilon_{xy}=\varepsilon_{yz}=\varepsilon_{zx}=\frac{1+\nu}{E}\cdot\tau_{xy}=\frac{\tau_{xy}}{2G}=0$

$\Rightarrow$ Cauchy: $u=Ax+C_1y+C_2z+C_3$

$v=Ay+C_4z+C_5x+C_6$

$w=Az+C_7x+C_8y+C_9$

$\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}=\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}=0$

ĐK1: Cố định tại O ($x=y=z=0 \Rightarrow u=v=w=0$)

ĐK2: Không quay tại O $\Rightarrow \frac{\partial u}{\partial y}=\frac{\partial u}{\partial z}=\frac{\partial v}{\partial z}=0$

$\Rightarrow$ $u=Ax$, $v=Ay$, $w=Az$

Điều kiện biên

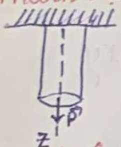
vtpt $\vec{n}(n_1,n_2)$, lực (F_x,F_y)

$$\begin{vmatrix}\sigma_x & \tau_{xy}\\ \tau_{xy} & \sigma_y\end{vmatrix}\begin{vmatrix}n_1\\ n_2\end{vmatrix}=\begin{vmatrix}F_x\\ F_y\end{vmatrix}$$

$\vec{n}\uparrow\uparrow\vec{p}\Rightarrow p(+)$

$n\uparrow\downarrow\vec{p}\Rightarrow p(-)$

Thanh trụ kéo bởi lực $\vec{P}$ ($\rho K=0$)



ĐKB: $\sigma_z=p=\frac{P}{S}$, $\sigma_x=\sigma_y=\tau_{ij}=0$

Hooke ngược: $\varepsilon_x=\varepsilon_y=-\frac{\nu}{E}p=A$

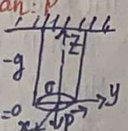
$\varepsilon_z=\frac{p}{E}=B$, $\varepsilon_{ij}=0$

$\Rightarrow$ Cauchy: $\Rightarrow u,v,w=\ldots$

$\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}=\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}=0$

ĐK1 + ĐK2 $\Rightarrow u=Ax$, $v=Ay$, $w=Bz$

Thanh trụ chịu trọng lực bản thân: $\vec{P}$



ĐKB: $\sigma_z=\frac{P}{S}=\frac{mg}{V/z}=\rho gz$

$\sigma_y=\sigma_x=\sigma_{ij}=0$

$K_z=-g$

PTCB: $\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \sigma_z}{\partial z}+\rho K_z=0$

Hooke ngược: $\varepsilon_x=\varepsilon_y=-\frac{\nu}{E}\cdot\rho gz=Az$, $\varepsilon_z=\frac{\rho gz}{E}$

Cauchy $\Rightarrow u=Azx+C_1y+C_2z+C_3$

$v=Azy+C_4z+C_5x+C_6$, $w=\frac{\rho g}{E}\cdot\frac{z^2}{2}+f(x)+g(y)+C_9$

$\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}=\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}=0$

$\Rightarrow\begin{cases}f'(x)=-(Ax+C_2)\\ g'(y)=-(Ay+C_4)\end{cases}\rightarrow\begin{cases}f(x)=-\frac{Ax^2}{2}-C_2x\\ g(y)=-\frac{Ay^2}{2}-C_4y\end{cases}$

ĐK1 + ĐK2 $\Rightarrow C_3=C_6=C_1=C_5=0$

$\frac{\partial u}{\partial z}=\frac{\partial v}{\partial z}=0\Rightarrow Ax+C_2=Ay+C_4=0$

$(x=y=0\Rightarrow C_2=C_4=0)$

$\Rightarrow u=Azx$, $v=Azy$, $w=\frac{\rho gz^2}{2E}-\frac{1}{2}A(x^2+y^2)$

ĐK1, $(x=y=z)=(0,0,L)$, $u=v=w=0$

$\Rightarrow C_2L+C_3=0$, $C_4L+C_6=0$

$\frac{\rho g}{E}\cdot\frac{L^2}{2}+C_9=0\Rightarrow C_9=-\frac{\rho g}{E}\frac{L^2}{2}$

$\Rightarrow u=Azx$, $v=Azy$, $w=\frac{\rho g}{2E}\left(z^2-L^2+\nu(x^2+y^2)\right)$

Bài toán trong hệ tọa độ cầu

Cauchy: $\varepsilon_r=\frac{\partial u_r}{\partial r}$

$\varepsilon_\theta=\frac{1}{r\sin\varphi}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}+\frac{u_\varphi}{r\tan\varphi}$

$\varepsilon_\varphi=\frac{1}{r}\cdot\frac{\partial u_\varphi}{\partial\varphi}+\frac{u_r}{r}$

$\varepsilon_{r\theta}=\frac{1}{2}\left(\frac{1}{r\sin\varphi}\frac{\partial u_r}{\partial\theta}+\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\right)$

$\varepsilon_{\theta\varphi}=\frac{1}{2}\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\varphi}+\frac{1}{r\sin\varphi}\frac{\partial u_\varphi}{\partial\theta}-\frac{1}{r\tan\varphi}u_\theta\right)$

$\varepsilon_{\varphi r}=\frac{1}{2}\left(\frac{\partial u_\varphi}{\partial r}-\frac{u_\varphi}{r}+\frac{1}{r}\frac{\partial u_r}{\partial\varphi}\right)$

PTCB: $\frac{\partial\sigma_r}{\partial r}+\frac{1}{r\sin\varphi}\cdot\frac{\partial\sigma_{r\theta}}{\partial\theta}+\frac{1}{r}\cdot\frac{\partial\sigma_{r\varphi}}{\partial\varphi}+\frac{1}{r}(2\sigma_r-\sigma_\theta-\sigma_\varphi-\tau_{r\varphi}\cot\varphi)+\rho K_r=\rho F_r$

$\frac{\partial\tau_{r\theta}}{\partial r}+\frac{1}{r\sin\varphi}\cdot\frac{\partial\sigma_\theta}{\partial\theta}+\frac{1}{r}\cdot\frac{\partial\tau_{\theta\varphi}}{\partial\varphi}+\frac{1}{r}(2\sigma_\theta\cot\varphi+3\tau_{r\theta})+\rho K_\theta=\rho F_\theta$

$\frac{\partial\tau_{r\varphi}}{\partial r}+\frac{1}{r\sin\varphi}\cdot\frac{\partial\tau_{\theta\varphi}}{\partial\theta}+\frac{1}{r}\cdot\frac{\partial\sigma_\varphi}{\partial\varphi}+\frac{1}{r}(3\tau_{r\varphi}+(\sigma_\varphi-\sigma_\theta)\cot\varphi)+\rho K_\varphi=\rho F_\varphi$

Hooke: $\sigma_r=\lambda\theta_e+2\mu\varepsilon_r$ — $\tau_{r\theta}=2\mu\varepsilon_{r\theta}\ldots$ $\theta_e=\frac{\partial u_r}{\partial r}+\frac{2u_r}{r}+\frac{1}{r\sin\varphi}\frac{\partial u_\theta}{\partial\theta}+\frac{u_\varphi}{r\tan\varphi}+\frac{1}{r}\cdot\frac{\partial u_\varphi}{\partial\varphi}$

* Cầu đx chịu tải đx $\Rightarrow u_r=u_r(r)$

$u_\varphi=u_\theta=0\Rightarrow\frac{\partial u_r}{\partial\varphi}=\frac{\partial u_r}{\partial\theta}=0$

Cầu rỗng áp suất trong, ngoài (p_1,p_2)

Cauchy: $\varepsilon_r=\frac{\partial u_r}{\partial r}$, $\varepsilon_\theta=\varepsilon_\varphi=\frac{u_r}{r}$, $\varepsilon_{r\theta}=\varepsilon_{\theta\varphi}=\varepsilon_{\varphi r}=0$, $\theta_e=\frac{\partial u_r}{\partial r}+\frac{2u_r}{r}$ $\Rightarrow$ Hooke: $\sigma_r=\lambda\left(\frac{\partial u_r}{\partial r}+\frac{2u_r}{r}\right)+2\mu\frac{\partial u_r}{\partial r}$, $\sigma_\theta=\sigma_\varphi=\lambda\left(\frac{\partial u_r}{\partial r}+\frac{2u_r}{r}\right)+2\mu\frac{u_r}{r}$

$\Rightarrow$ PTCB: $\frac{d\sigma_r}{dr}+\frac{2}{r}(\sigma_r-\sigma_\varphi)=0\Rightarrow\lambda\left(\frac{d^2u_r}{dr^2}+2\cdot\frac{\frac{du_r}{dr}\cdot r-u_r}{r^2}\right)+2\mu\frac{d^2u_r}{dr^2}+\frac{2}{r}2\mu\left(\frac{du_r}{dr}-\frac{u_r}{r}\right)\Rightarrow(\lambda+2\mu)\left(\frac{d^2u_r}{dr^2}+\frac{2}{r}\frac{du_r}{dr}-\frac{1}{r^2}\cdot 2u_r\right)=0$

$\Rightarrow$ PTVP: $\frac{d^2u_r}{dr^2}+\frac{2}{r}\frac{du_r}{dr}-\frac{2u_r}{r^2}=0\Rightarrow$ chọn nghiệm $u_r=Ar+\frac{B}{r^2}$ $\Rightarrow\theta_e=A-\frac{2B}{r^3}+2\left(A+\frac{B}{r^3}\right)=3A\Rightarrow\sigma_r=(3\lambda+2\mu)A-\frac{4B\mu}{r^3}$

$\sigma_\varphi=\sigma_\theta=(3\lambda+2\mu)A+\frac{2B\mu}{r^3}$

ĐKB: $\begin{cases}\sigma_r|_{r=a}=(3\lambda+2\mu)A-\frac{4\mu B}{a^3}=-p_1\\ \sigma_r|_{r=b}=(3\lambda+2\mu)A-\frac{4\mu B}{b^3}=-p_2\end{cases}\Rightarrow\begin{cases}A=\frac{p_2b^3-p_1a^3}{(3\lambda+2\mu)(a^3-b^3)}\\ B=\frac{a^3b^3(p_2-p_1)}{4\mu(a^3-b^3)}\end{cases}\Rightarrow u_r$

Cầu không đồng chất áp suất ngoài p_2 ($a:\lambda_1,\mu_1$, $b:\lambda_2,\mu_2$)

$\Rightarrow$ PTVP: $\frac{d^2u_r}{dr^2}+\frac{2}{r}\frac{du_r}{dr}-\frac{2}{r^2}u_r=0$ 2 nghiệm $\rightarrow u_r^{(1)}=A_1r$, $u_r^{(2)}=A_2r+\frac{B_2}{r^2}$

Miền (1): $0\le r\le a$ nghiệm: $u_r^{(1)}=A_1r\Rightarrow\theta_e=A_1+2A_1=3A_1\Rightarrow\sigma_r=\lambda_1\theta_e+2\mu A_1=(3\lambda_1+2\mu_1)A_1$

ĐKB: $\sigma_r|_{r=a}=p_1\Rightarrow(3\lambda_1+2\mu_1)A_1=p_1\Rightarrow u_r^{(1)}=\frac{p_1r}{3\lambda_1+2\lambda_1}$

Miền (2): $a\le r\le b$ nghiệm $u_r^{(2)}=A_2r+\frac{B_2}{r^2}\Rightarrow\sigma_r=(3\lambda_2+2\mu_2)A_2-\frac{4B\mu_2}{r^3}$, σ_θ

ĐKB: $\begin{cases}\sigma_r|_{r=a}=p_1\\ \sigma_r|_{r=b}=p_2\end{cases}\Rightarrow\begin{cases}A_2=\frac{-p_2b^3+p_1a^3}{(3\lambda_2+2\mu_2)(a^3-b^3)}\\ B_2=\frac{a^3b^3(p_1-p_2)}{4\mu_2(a^3-b^3)}\end{cases}\Rightarrow u_r^{(2)}=A_2r+\frac{B_2}{r^2}$

$u_r^{(1)}|_{r=a}=u_r^{(2)}|_{r=a}$

$\Rightarrow\frac{p_1a}{3\lambda_1+2\mu_1}=\frac{p_1a^3-p_2a^3}{(3\lambda_2+2\mu_2)(a^3-b^3)}r+\frac{1}{r^2}\frac{a^3b^3(p_1-p_2)}{4\mu_2(a^3-b^3)}$

$\Rightarrow p_1=\frac{p_2b^3(6\mu_2+3\lambda_2)(2\mu_1+3\lambda_1)}{(2\mu_1+3\lambda_1)(4\mu_2a^3+(2\mu_2+3\lambda_2)b^3)+(2\mu_2+3\lambda_2)4\mu_2(b^3-a^3)}$

$\Rightarrow$ US-CV u_r $\Rightarrow A_1$

Cầu không đồng chất áp suất ngoài + nhiệt độ ΔT

$\Rightarrow$ PTVP: $\frac{d^2u_r}{dr^2}+\frac{2}{r}\frac{du_r}{dr}-\frac{2}{r^2}u_r=0$

Miền (2): $a\le r\le b$: $\sigma_r=(3\lambda_2+2\mu_2)A_2-\frac{4\mu B_2}{r^3}$

($\beta=(3\lambda_2+2\mu_2)\alpha_2\Delta T$: áp suất do nhiệt độ)

ĐKB: $\begin{cases}\sigma_r|_{r=a}=(3\lambda_2+2\mu_2)A_2-\frac{4\mu B_2}{a^3}=p-(3\lambda_2+2\mu_2)\alpha_2\Delta T\\ \sigma_r|_{r=b}=(3\lambda_2+2\mu_2)A_2-\frac{4\mu B_2}{b^3}=p_2-(3\lambda_2+2\mu_2)\alpha_2\Delta T\end{cases}$

$\Rightarrow-4\mu B_2\left(\frac{1}{a^3}-\frac{1}{b^3}\right)=p-p_2\Rightarrow B_2=\frac{a^3b^3(p-p_2)}{4\mu_2(a^3-b^3)}\Rightarrow A_2=\frac{1}{3\lambda_2+2\mu_2}\cdot\frac{pa^3-p_2b^3}{a^3-b^3}-\alpha_2\Delta T$

Miền (1): $0\le r\le a$. $\sigma_r=(3\lambda_1+2\mu_1)A_1=p-(3\lambda_1+2\mu_1)\alpha_1\Delta T$ (đKb) $\Rightarrow A_1=\frac{p}{3\lambda_1+2\mu_1}-\alpha_1\Delta T=\frac{\alpha_1\Delta T\,3K_1(3K_2b^3+4\mu_2a^3)+3K_2 4\mu_2\alpha_2\Delta T(b^3-a^3)-(3K_2+4\mu_2)p_2b^3}{3K_1(3K_2b^3+4\mu_2a^3)+3K_2 4\mu_2(b^3-a^3)}$

$u_r^{(1)}|_{r=a}=u_r^{(2)}|_{r=a}\Rightarrow\left(\frac{p}{3\lambda_1+2\mu_1}-\alpha_1\Delta T\right).a=\left(\frac{1}{3\lambda_2+2\mu_2}\frac{pa^3-p_2b^3}{a^3-b^3}-\alpha_2\Delta T\right).a+\frac{a^3b^3(p-p_2)}{4\mu_2a^2(a^3-b^3)}$

$K_i=\lambda_i+\frac{2}{3}\mu_i$ $(i=1,2)$

$\Rightarrow p=-3K_1\cdot\frac{3K_2 4\mu_2\alpha_2\Delta T(b^3-a^3)-3K_2 4\mu_2\alpha_1\Delta T(b^3-a^3)-(3K_2+4\mu_2)p_2b^3}{3K_1(3K_2b^3+4\mu_2a^3)+3K_2\cdot 4\mu_2(b^3-a^3)}=-3K_1\cdot\frac{3K_2 4\mu_2\Delta T(b^3-a^3)(\alpha_2-\alpha_1)-(3K_2+4\mu_2)p_2b^3}{3K_1(3K_2b^3+4\mu_2a^3)+3K_2 4\mu_2(b^3-a^3)}$

Bài toán trong hệ tọa độ trụ

Cauchy: $\varepsilon_r=\frac{\partial u_r}{\partial r}$

$\varepsilon_\theta=\frac{1}{r}\cdot\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}$

$\varepsilon_z=\frac{\partial u_z}{\partial z}$

$\varepsilon_{r\theta}=\frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta}+\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\right)$

$\varepsilon_{\theta r}=\frac{1}{2}\left(\frac{\partial u_\theta}{\partial z}-\frac{1}{r}\cdot\frac{\partial u_z}{\partial\theta}\right)$

$\varepsilon_{rz}=\frac{1}{2}\left(\frac{\partial u_z}{\partial r}-\frac{1}{r}\cdot\frac{\partial u_r}{\partial z}\right)$

PTCB: $\frac{\partial\sigma_r}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta}+\frac{\partial\sigma_{rz}}{\partial z}+\frac{\sigma_r-\sigma_\theta}{r}+\rho K_r=\rho F_r$

$\frac{\partial\sigma_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta}+\frac{\partial\sigma_{\theta z}}{\partial z}+\frac{2\sigma_{r\theta}}{r}+\rho K_\theta=\rho F_\theta$

$\frac{\partial\sigma_{rz}}{\partial r}+\frac{1}{r}\cdot\frac{\partial\sigma_{rz}}{\partial\theta}+\frac{\partial\sigma_z}{\partial z}+\frac{2\sigma_{rz}}{r}+\rho K_z=\rho F_z$

Hooke: $\sigma_r=\lambda\theta_e+2\mu\varepsilon_r\ldots$

$\sigma_{r\theta}=2\mu\varepsilon_{r\theta}$

$\theta_e=\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{1}{r}\cdot\frac{\partial u_\theta}{\partial\theta}+\frac{\partial u_r}{\partial z}$

Trụ đx chịu tải đx $\Rightarrow u_r=u_r(r)$

$u_\theta=u_z=0$, $\frac{\partial u_r}{\partial\theta}=\frac{\partial u_r}{\partial z}=0$

Trụ

Trụ rỗng chịu áp suất trong, ngoài (p_1,p_2)

Cauchy: $\varepsilon_r=\frac{\partial u_r}{\partial r}$, $\varepsilon_\theta=\frac{u_r}{r}$, $\varepsilon_z=\varepsilon_{r\theta}=\varepsilon_{\theta z}=\varepsilon_{zr}=0$, $\theta_e=\frac{\partial u_r}{\partial r}+\frac{u_r}{r}\Rightarrow$ Hook: $\sigma_r=\lambda\left(\frac{\partial u_r}{\partial r}+\frac{u_r}{r}\right)+2\mu\frac{\partial u_r}{\partial r}$, $\sigma_\theta=\lambda\left(\frac{\partial u_r}{\partial r}+\frac{u_r}{r}\right)+2\mu\frac{u_r}{r}$, $\sigma_z=\lambda\left(\frac{\partial u_r}{\partial r}+\frac{u_r}{r}\right)$

PTCB: $\frac{d\sigma_r}{dr}+\frac{\sigma_r-\sigma_\theta}{r}=0\Rightarrow\lambda\left(\frac{d^2u_r}{dr^2}+\frac{\frac{du_r}{dr}\cdot r-u_r}{r^2}\right)+2\mu\frac{d^2u_r}{dr^2}+\frac{2\mu}{r}\left(\frac{du_r}{dr}-\frac{u_r}{r}\right)=0\Rightarrow(\lambda+2\mu)\left(\frac{d^2u_r}{dr^2}+\frac{1}{r}\cdot\frac{du_r}{dr}-\frac{u_r}{r^2}\right)=0$

$\Rightarrow$ PTVP: $\frac{d^2u_r}{dr^2}+\frac{1}{r}\cdot\frac{du_r}{dr}-\frac{u_r}{r^2}=0\Rightarrow$ chọn nghiệm: $u_r=Ar+\frac{B}{r}\Rightarrow\theta_e=A-\frac{B}{r^2}+A+\frac{B}{r^2}=2A\Rightarrow\sigma_r=2(\lambda+\mu)A-\frac{2\mu B}{r^2}$ $\sigma_\theta=2(\lambda+\mu)A+\frac{2\mu B}{r^2}$

ĐKB: $\begin{cases}\sigma_r|_{r=a}=-p_1\\ \sigma_r|_{r=b}=-p_2\end{cases}\Rightarrow\begin{cases}2(\lambda+\mu)A-\frac{2\mu B}{a^2}=-p_1\\ 2(\lambda+\mu)A-\frac{2\mu B}{b^2}=-p_2\end{cases}\Rightarrow A=\frac{p_2b^2-p_1a^2}{2(\lambda+\mu)(a^2-b^2)}$, $B=\frac{(p_2-p_1)a^2b^2}{2\mu(a^2-b^2)}\Rightarrow u_r,\varepsilon,\sigma$

$\varepsilon_r=A-\frac{B}{r^2}$, $\varepsilon_\theta=A+\frac{B}{r^2}$

Trụ không đồng chất chịu áp suất ngoài p_2 - PTVP: $\frac{d^2u_r}{dr^2}+\frac{1}{r}\cdot\frac{du_r}{dr}-\frac{u_r}{r^2}=0 \Rightarrow$ no TQ: $u_r(1)=A_1r$ $\quad u_r(2)=A_2r+\frac{B_2}{r}$

Miền (1): $0\le r\le a$: $\sigma_r=2(\lambda_1+\mu_1)A_1=P$ (ĐKB) $\Rightarrow A_1=P/2(\lambda_1+\mu_1) \Rightarrow u_r^{(1)}=\frac{Pr}{2(\lambda_1+\mu_1)}$

Miền (2): $a\le r\le b$: $u_r^{(2)}=A_2r+\frac{B_2}{r} \Rightarrow \sigma_r=2(\lambda_2+\mu_2)A_2-\frac{2\mu_2B_2}{r^2}$

ĐKB: $\begin{cases}\sigma_r|_{r=a}=P\\ \sigma_r|_{r=b}=P_2\end{cases} \Rightarrow \begin{cases}A_2=\frac{Pa^2-P_2b^2}{2(\lambda_2+\mu_2)(a^2-b^2)}\\ B_2=\frac{(P-P_2)a^2b^2}{2\mu_2(a^2-b^2)}\end{cases}$

$u_r^{(1)}=u_r^{(2)}$ (r=a) $\Rightarrow \frac{(Pa^2-P_2b^2)a}{2(\lambda_2+\mu_2)(a^2-b^2)}+\frac{(P-P_2)a^2b^2}{2\mu_2(a^2-b^2)a}=\frac{Pa}{2(\lambda_1+\mu_1)}$

$\Rightarrow P=\frac{b^2P_2(\lambda_1+\mu_1)(\lambda_2+2\mu_2)}{\mu_2[(\lambda_1+\mu_1)-(\lambda_2+\mu_2)]a^2+(\lambda_2+\mu_2)(\lambda_1+\mu_1+\mu_2)} \rightarrow A_1 \Rightarrow$ ƯS, CV

Bài toán biến dạng phẳng: Trạng thái biến dạng - mọi điểm chuyển động song song với một mặt phẳng cố định, mọi điểm nằm trên cùng một đg thẳng bất kỳ trực giao với mp cố định sẽ có chuyển dịch như nhau. Cách đặt bài toán.

$u=u(x,y)$, $v=v(x,y)$, $w=0 \Rightarrow$ Cauchy: $\varepsilon_x=\frac{\partial u}{\partial x}$, $\varepsilon_y=\frac{\partial v}{\partial y}$, $\varepsilon_{xy}=\frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)$, $\varepsilon_z=\varepsilon_{xz}=\varepsilon_{zx}=0$

PTTTBD: $\frac{\partial^2\varepsilon_x}{\partial y^2}+\frac{\partial^2\varepsilon_y}{\partial x^2}=2\frac{\partial^2\varepsilon_{xy}}{\partial x\partial y}$

Định luật Hooke: $\varepsilon_x=\frac{1}{E_1}(\sigma_x-\nu_1\sigma_y)$, $\varepsilon_y=\frac{1}{E_1}(\sigma_y-\nu_1\sigma_x)$ $\varepsilon_{xy}=\frac{\sigma_{xy}}{2G}=\frac{1+\nu_1}{E_1}\cdot\sigma_{xy}$, $\varepsilon_z=0$, $\sigma_z=\nu(\sigma_x+\sigma_y)\neq0$

$\left(E_1=\frac{E}{1-\nu^2}, \nu_1=\frac{\nu}{1-\nu}\right)$ thành phần σ_z đảm bảo việc tồn tại biến dạng phẳng

PTVP cân bằng: $\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+X=0$ - $\frac{\partial\sigma_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+Y=0$.

Bài toán ứng suất phẳng: trạng thái ứng suất trong 1 vật thể sao cho tại các tiết diện song song với một mặt phẳng cố định ứng suất bằng 0, còn tại các điểm khác, ưs không phụ thuộc vào k/c từ điểm đang xét tới mp cố định.

$\sigma_z=\tau_{yz}=\tau_{zx}=0$, $\sigma_x,\sigma_y,\tau_{xy}\in(x,y)$ $\quad \varepsilon_x=\frac{1}{E}(\sigma_x-\nu\sigma_y)$ ~~$\tau_{xy}=2\mu\varepsilon_{xy}$~~ $\varepsilon_{xy}=\frac{1+\nu}{E}\cdot\tau_{xy}$

ĐL Hooke: $\sigma_x=\lambda\theta+2\mu\varepsilon_x$, $\sigma_y=\lambda\theta+2\mu\varepsilon_y$, $\tau_{xy}=2\mu\varepsilon_{xy}$ $\quad \varepsilon_y=\frac{1}{E}(\sigma_y-\nu\sigma_x)$ $\quad \varepsilon_z=\frac{-\nu}{E}(\sigma_x+\sigma_y)$

Hàm ƯS Airy: Lực khối $\rho K=0$, ptcb trong TH biến dạng phẳng:

$\begin{cases}\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}=0\\ \frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\sigma_y}{\partial y}=0\end{cases} \Rightarrow$ ∃ tồn tại hàm $A(x,y)$ và $B(x,y)$ t/m: $\begin{cases}\sigma_x=\frac{\partial A}{\partial y}, & \tau_{xy}=-\frac{\partial A}{\partial x}\\ \sigma_y=\frac{\partial B}{\partial x}, & \tau_{xy}=-\frac{\partial B}{\partial y}\end{cases} \Rightarrow -\frac{\partial A}{\partial x}=-\frac{\partial B}{\partial y}$

$\Rightarrow$ tồn tại hàm $U(x,y)$: $A=\frac{\partial U}{\partial y}$ - $B=\frac{\partial U}{\partial x}$

Biểu diễn các thành phần ưs qua hàm $U(x,y)$: $\sigma_x=\frac{\partial^2U}{\partial y^2}$, $\sigma_y=\frac{\partial^2U}{\partial x^2}$, $\tau_{xy}=-\frac{\partial^2U}{\partial x\partial y}$

Điều kiện của hàm ưs Airy: $\frac{\partial^4U}{\partial x^4}+2\frac{\partial^4U}{\partial x^2\partial y^2}+\frac{\partial^4U}{\partial y^4}=0$ (pt Max-well - pt lưỡng điều hoà)

Mối quan hệ giữa nội lực và ưs: $N_x=\int_A\sigma_x dA$, $Q_y=\int_A\tau_{xy}dA$, $M_z=\int_A\sigma_x y\,dA$ (dA: phân tố S bao quanh điểm; N_x: lực dọc trục (+ hướng ra ngoài); Q_y: lực cắt (+ quay cùng chiều kđh); M_z: moment quán tính theo z)

Thanh bị uốn $M_0=Kh$ - $\varphi=Ay^3$ $\quad M_0=M_z$

ĐK hàm ưs Airy: $\Delta\Delta F=0 \Rightarrow \sigma_x=6Ay$, $\sigma_y=\tau_{xy}=0$

Công thức nội lực: $M_z=\int_{-h/2}^{h/2}6Ay^2dy=2Ay^3\Big|_{-h/2}^{h/2}=4A\left(\frac{h}{2}\right)^3=\frac{Ah^3}{2} \Rightarrow \frac{Ah^3}{2}=Kh \Rightarrow A=\frac{2K}{h^2} \Rightarrow \sigma_x=\frac{12Ky}{h^2}$, $\sigma_y=\tau_{xy}=0$

Hai gối tựa: $\varphi(x,y)=\frac{d}{6}y^3\left(x^2-\frac{y^2}{5}\right)+\frac{b}{2}x^2y+\frac{K}{6}y^3+\frac{a}{2}x^2 \Rightarrow \sigma_x=\frac{\partial^2\varphi}{\partial y^2}=dyx^2-\frac{2}{3}dy^3+Ky$

$\frac{\partial^4\varphi}{\partial x^2\partial y^2}=2dy$, $\frac{\partial^4\varphi}{\partial y^4}=-4dy \Rightarrow$ t/m HƯS Airy $\quad \sigma_y=\frac{\partial^2\varphi}{\partial x^2}=\frac{d}{3}y^3+by+a$ $\quad \tau_{xy}=-dy^2x-bx$

ĐKB: $y=-\frac{h}{2}$ $n(0,-1)$ $F(0,-q) \Rightarrow \begin{cases}\sigma_y=-q\\ \tau_{xy}=0\end{cases}$ $\quad y=\frac{h}{2} \Rightarrow \sigma_y=\tau_{xy}=0$

PT nội lực. tại $x=\frac{l}{2}$, $M_z=\int_{-h/2}^{h/2}\left[dx^2y^2-\frac{2}{3}dy^4+Ky^2\right]dy$

$\begin{cases}-\frac{dh^3}{24}-\frac{bh}{2}+a=-q\\ -\left(\frac{dh^2}{4}+b\right)x=0 \Rightarrow \frac{dh^2}{4}+b=0\\ \frac{dh^3}{24}+\frac{bh}{2}+a=0\end{cases} \Rightarrow \begin{cases}a=-\frac{q}{2}\\ b=\frac{3q}{2h}\\ d=-\frac{6q}{h^3}\end{cases}$

$=d\frac{l^2}{4}\cdot\frac{y^3}{3}-\frac{2}{15}dy^5+K\frac{y^3}{3}\Big|_{-h/2}^{h/2}=2\left[d\frac{l^2}{4}\cdot\frac{h^3}{8.3}-\frac{2}{15}d\frac{h^5}{32}+K\frac{h^3}{8.3}\right]=0$

$\Rightarrow K\frac{h^3}{24}=\frac{dh^5}{16.15}-\frac{dl^2h^3}{96} \Rightarrow K=d\left(\frac{h^2}{10}-\frac{l^2}{4}\right)=\frac{6q}{h^3}\left(\frac{l^2}{4}-\frac{h^2}{10}\right)$

Dầm chịu trọng lượng bản thân $\varphi=\frac{1}{6}\rho g(y^3+3cx^2)$ $\quad$ ∃ $A(x,y)$ và $B(x,y)$ t/m.

Lực khối $\neq0 \Rightarrow$ PTCB: $\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}=0$, $\frac{\partial\sigma_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\rho g=0$ $\quad \begin{cases}\sigma_x=\frac{\partial A}{\partial y}, & \tau_{xy}=-\frac{\partial A}{\partial x}\\ \tau_{xy}=-\frac{\partial B}{\partial y} \Rightarrow \frac{\partial\sigma_y}{\partial y}=\frac{\partial^2B}{\partial x\partial y}-\rho g \Rightarrow \sigma_y=\frac{\partial B}{\partial x}-\rho gy\end{cases}$

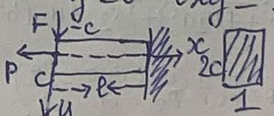
$\Rightarrow \frac{\partial A}{\partial x}=\frac{\partial B}{\partial y} \Rightarrow \exists F$ t/m: $A=\frac{\partial F}{\partial y}$, $B=\frac{\partial F}{\partial x}$

$\Rightarrow \sigma_x=\frac{\partial^2F}{\partial y^2}=\rho gy$ $\quad \sigma_y=\frac{\partial^2F}{\partial x^2}-\rho gy=\rho gc-\rho gy$, $\tau_{xy}=-\frac{\partial^2F}{\partial x\partial y}=0$

Bài toán ngược: $\varphi=\frac{3F}{4C}\left(xy-\frac{xy^3}{3c^2}\right)+\frac{P}{4C}y^2$ $\quad \Delta\Delta F=0 \Rightarrow$ HƯS Airy $\Rightarrow \sigma_x=\frac{\partial^2\varphi}{\partial y^2}=-\frac{3}{2}\frac{F}{C^3}xy+\frac{P}{2C}$

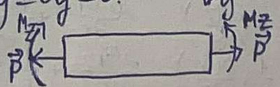
Công thức nội lực: $N_x=\int_{-c}^{c}\sigma_x dy=P>0$, $Q_y=\int_{-c}^{c}\tau_{xy}dy=-F<0$ $\quad \sigma_y=0$ $\quad \tau_{xy}=-\frac{3F}{4C}+\frac{3F}{4C^3}\cdot y^2$

$M_z=\int_{-c}^{c}\sigma_x y\,dy=-Fx<0$ Tại $x=0$, $M_z=0$, $x=l$, $M_z=-Fl$



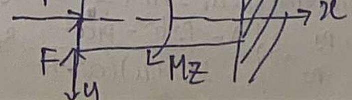
$\varphi=\frac{1}{2}Sy\left(1-\frac{1}{6}\frac{y^2}{c^2}\right)+\frac{Py^2}{4C} \Rightarrow \sigma_x=-\frac{S}{2c^2}y+\frac{P}{2C}$, $\tau_{xy}=\sigma_y=0$.

$N_x=P>0$, $Q_y=0$, $M_z=-\frac{Sc}{3}=$ const.



$\varphi=\frac{-F}{d^3}xy^2(3d-2y)-\frac{Py^2}{2d} \Rightarrow \sigma_x=-\frac{F}{d^3}3dx2+\frac{12F}{d^3}xy-\frac{P}{d}$ $\quad \sigma_y=0$ $\quad \tau_{xy}=\frac{6Fy}{d^2}-\frac{6Fy^2}{d^3}$

$N_x=-P<0$, $Q_y=F>0$, $M_z=Fx-\frac{Pd}{2}$, $x=0$: $M_z=-\frac{Pd}{2}$ $\quad x=l$: $M_z=Fl-\frac{Pd}{2}$



Bài toán phẳng trong hệ tọa độ cực: Biến dạng phẳng: $u_r = u_r(r,\theta)$, $u_\theta = u_\theta(r,\theta)$, $u_z = 0$

Cauchy: $\varepsilon_r = \frac{\partial u_r}{\partial r}$, $\varepsilon_\theta = \frac{1}{r}\cdot\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}$, $\varepsilon_{r\theta} = \frac{1}{r}\cdot\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}$ $\varepsilon_z = \varepsilon_{rz} = \varepsilon_{\theta z} = 0$

Hooke: $\sigma_r = \lambda\theta + 2\mu\varepsilon_r$, $\sigma_z = \lambda\theta$, $\sigma_\theta = \lambda\theta + 2\mu\varepsilon_\theta$, $\tau_{r\theta} = 2\mu\varepsilon_{r\theta}$, $\tau_{rz} = \tau_{\theta z} = 0$ $\theta = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\cdot\frac{\partial u_\theta}{\partial\theta}$

PTCB: $\frac{\partial\sigma_r}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{\sigma_r - \sigma_\theta}{r} + \rho K_r = \rho F_r$ - $\frac{\partial\tau_{r\theta}}{\partial r} + \frac{1}{r}\cdot\frac{\partial\sigma_\theta}{\partial\theta} + \frac{2\tau_{r\theta}}{r} + \rho K_\theta = \rho F_\theta$

MQH hàm ứs với trường ứng suất:

$\sigma_r = \frac{1}{r}\cdot\frac{\partial F}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2 F}{\partial\theta^2}$, $\sigma_\theta = \frac{\partial^2 F}{\partial r^2}$

$\sigma_{r\theta} = \frac{1}{r^2}\cdot\frac{\partial F}{\partial\theta} - \frac{1}{r}\cdot\frac{\partial^2 F}{\partial r\partial\theta}$

Kiểm tra đk hàm ứng suất:

$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2 F}{\partial\theta^2}\right) = 0$

Hàm ứs: $F = f_1(r)\sin\theta + f_2(r)\cos\theta$ $f_1(r) = Ar^3 + Br\ln r + Cr + \frac{D}{r}$ $f_2(r) = Er$

$\Rightarrow \sigma_r = \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial F^2}{\partial\theta} = \left(2Ar + \frac{B}{r} - \frac{2D}{r^3}\right)\sin\theta$ $\sigma_\theta = \frac{\partial^2 F}{\partial r^2} = \left(6Ar + \frac{B}{r} + \frac{2D}{r^3}\right)\sin\theta$ $\tau_{r\theta} = \frac{1}{r^2}\frac{\partial F}{\partial\theta} - \frac{1}{r}\frac{\partial^2 F}{\partial r\partial\theta}$ $= \left(-2Ar - \frac{B}{r} + \frac{2D}{r^3}\right)\cos\theta$

ĐKB ở bKính trong (r=a), bán kính ngoài (r=b) ⇒ σ_r, σ_θ - hàm phân bố theo hàm sin, $\tau_{r\theta}$ - hàm pbố theo hàm cos

KL: Vật thể chịu lực theo hướng r, θ - hàm sin - vật thể chịu tải trọng tiếp xúc - hàm cos.

1/4 hình vành khăn ⇒ chọn hàm ứs: $\varphi = f(r)\sin\theta$

Thay vào pt điều hoà levy $\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\left(\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} - \frac{f}{r^2}\right) = 0$ ⇒ NTQ: $f(r) = C_1 r^3 + \frac{C_2}{r} + C_3 r + C_4 r\ln r$

$\Rightarrow \varphi = \left(C_1 r^3 + \frac{C_2}{r} + C_3 r + C_4 r\ln r\right)\sin\theta$

$\Rightarrow \sigma_r = \left(2C_1 r - \frac{2C_2}{r^3} + \frac{C_4}{r}\right)\sin\theta$, $\sigma_\theta = \left(6C_1 r + \frac{2C_2}{r^3} + \frac{C_4}{r}\right)\sin\theta$

$\tau_{r\theta} = \left(-2C_1 r + \frac{2C_2}{r^3} - \frac{C_4}{r}\right)\cos\theta$. ĐKB: $\sigma_r|_{r=a} = \sigma_r|_{r=b} = 0 \Rightarrow \begin{cases} 2C_1 a - \frac{2C_2}{a^3} + \frac{C_4}{a} = 0 \\ 2C_1 b - \frac{2C_2}{b^3} + \frac{C_4}{b} = 0 \end{cases} \Rightarrow 2C_1(a^2 - b^2) = 2C_2\left(\frac{1}{a^4} - \frac{1}{b^4}\right)$

$\Rightarrow C_2 = -C_1 a^2 b^2$

$\Rightarrow C_4 = -2C_1(a^2 + b^2)$

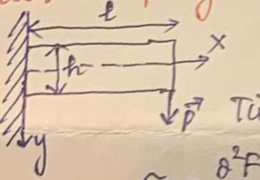
Lực cắt: $P = \int_a^b \tau_{r\theta}\,dr = \int_a^b\left(-2C_1 r + \frac{2C_2}{r^3} - \frac{C_4}{r}\right)\cos\theta\,dr$

$= -C_1 r^2 - \frac{C_2}{r^2} - C_4\ln r\Big|_a^b = -C_1(b^2 - a^2) - C_2\left(\frac{1}{b^2} - \frac{1}{a^2}\right) - C_4\ln\frac{b}{a} = C_1(a^2 - b^2) + C_1(a^2 - b^2) + 2C_1(a^2 + b^2)\ln\frac{b}{a}$

$\Rightarrow C_1 = \frac{P}{2[a^2 - b^2 + (a^2 + b^2)\ln\frac{b}{a}]}$ ⇒ C_2, C_4 ⇒ $\sigma_r = 2C_1\left(r + \frac{a^2b^2}{r^3} - \frac{a^2 + b^2}{r}\right)\sin\theta$ $\sigma_\theta = 2C_1\left(3r - \frac{a^2b^2}{r^3} - \frac{a^2 + b^2}{r}\right)$

$\tau_{r\theta} = 2C_1\left(-r - \frac{a^2b^2}{r^3} + \frac{a^2 + b^2}{r}\right)$

Uốn dầm phẳng công-xôn



Xem trạng thái ứng suất phẳng, độ dày $\delta \ll h$ ⇒ $\sigma_x = \frac{\partial^2 F}{\partial y^2}$, $\sigma_y = \frac{\partial^2 F}{\partial x^2}$, $\tau_{xy} = -\frac{\partial^2 F}{\partial x\partial y}$

Đặt $\sigma_x = C_0 y + C_1 x \Rightarrow F(x,y) = \frac{C_0 y^3}{6} + \frac{C_1 x y^2}{2} + y f_1(x) + f_2(x)$

Từ pt $\Delta\Delta F = 0 \Rightarrow y\frac{d^4 f_1(x)}{dx^4} + \frac{d^4 f_2(x)}{dx^4} = 0 \Rightarrow \begin{cases} f_1(x) = C_2 x^3 + C_3 x^2 + C_4 x + C_5 \\ f_2(x) = C_6 x^3 + C_7 x^2 + C_8 x + C_9 \end{cases}$

$\Rightarrow \sigma_y = \frac{\partial^2 F}{\partial x^2} = 6(C_2 y + C_6)x + 2(C_3 y + C_7)$ $\tau_{xy} = \frac{-C_1 y^2}{2} - 3C_2 x^2 - 2C_3 x - C_4$

ĐKB: $y = \pm\frac{h}{2}$ t/m bt phẳng ⇒ $\sigma_y = \tau_{xy} = 0$ ⇒ $\begin{cases} \sigma_x = \frac{P(l - x)}{J_Z}\cdot y & (J_Z = \frac{h^3\delta}{12}) \\ \sigma_y = 0, \ \tau_{xy} = \frac{-P}{2J_Z}\left(\frac{h^2}{4} - y^2\right) \end{cases}$

$x = l$, $\sigma_x = 0$, $\int_{-h/2}^{h/2}\tau_{xy}\,dy = -P$

Áp dụng Hooke - Cauchy ⇒ $u = \frac{-P}{EJ}\left(lxy - \frac{x^2 y}{2}\right) + \frac{P}{EJ}f_3(y)$, $v = \frac{-\nu P}{EJ}\left(\frac{ly^2}{2} - \frac{x^2 y}{2}\right) + \frac{P}{EJ}f_4(y)$

ĐKB: tại gốc tọa độ (0,0,0) ⇒ $u = v = 0$, $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = 0$ (không cvị + quay)

⇒ Các thành phần chuyển vị: $u = \frac{P}{EJ}\left[-\left(l - \frac{x}{2}\right)xy - \frac{(2+\nu)y^3}{6} + \frac{(1+\nu)h^2 y}{4}\right]$ $v = \frac{P}{EJ}\left[\frac{\nu(l - x)y^2}{2} + \frac{lx^2}{2} - \frac{x^3}{6}\right]$

Bài toán xoắn thanh tròn (tiết diện tròn bK b chịu lực ngoài td ở 2 đáy, moment xoắn M)

Giả thiết tiết diện phẳng - bán kính phẳng. Tại mỗi điểm của tiết diện có ứng suất tiếp T_s thẳng góc với bán kính vectơ của điểm. β là góc quay giữa 2 mặt cắt ⇒ biến dạng trượt: $\gamma = \beta r$ ⇒ tp ứs tiếp $T_s = G\gamma = \beta G r$

Trạng thái ứs có dạng: $\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0$, $\tau_{zx} = -T_s\cdot\frac{y}{r} = -G\beta y$, $\tau_{yz} = +T_s\frac{x}{r} = \pm G\beta r$

Trạng thái ứs phải thoả mãn pt tương thích Beltrami do đó: $K_x = K_y = K_z = 0$

ĐKB: + Tại mặt bên tương ứng các cosin chỉ phương $(0, \frac{y}{r}, \frac{x}{r})$ thay vào biểu thức của ứs sau đó thay vào hpt đkb ta được $\Sigma x = \Sigma y = \Sigma z = 0$ t/m đk mặt bên không có lực tác dụng

+ Trên 2 mặt đáy: $n = \pm 1$, $m = l = 0$, cho ta $\Sigma x = \tau_{xz} = -G\beta y$, $\Sigma y = \tau_{yz} = G\beta x$, $\Sigma z = 0$ ⇒ chỉ có lực tiếp tuyến tdụng

Hợp lực của chúng chiếu lên các trục bằng 0: $\iint_S \tau_{xz}\,dS = -\iint_S G\beta y\,dS = 0$, $\iint_S \tau_{yz}\,dS = \iint_S G\beta x\,dS = 0$

Moment xoắn với trục z: $M = \iint_S(x\tau_{yz} - y\tau_{zx})\,dS = G\beta\iint_S(x^2 + y^2)\,dS = G\beta J_p$ $J_p = \frac{\pi b^4}{2}$: moment quán tính cực của mặt S

$\Rightarrow \beta = \frac{M}{GJ_p}$: GJ_p gọi là độ cứng khi xoắn. Thay các ứs vào đl Hooke và pt Cauchy

$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial w}{\partial z} = 0$, $\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0$, $\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = \beta x$, $\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = -\beta y$

Tích phân ta được: $u = -\beta yz + C_1 y - C_2 z + C_3$, $w = C_2 x - C_4 y + C_6$

$v = \beta xz - C_1 x + C_4 z + C_5$

ĐKB: tại gốc toạ độ: không chuyển dịch, không xoay: $u = v = w = 0$, $\frac{\partial v}{\partial x} = \frac{\partial w}{\partial x} = \frac{\partial w}{\partial y} = 0$

Khi đó: $u = -\beta yz$, $v = \beta xz$, $w = 0$

Bài toán uốn thuần tuý của dầm – Mọi tiết diện của dầm có trạng thái US – BD như nhau.

Giả thiết biến dạng phẳng – trạng thái US: $\sigma_x = \sigma_y = \tau_{xy} = \tau_{yz} = \tau_{zx} = 0$, $\sigma_z = -E\frac{x}{R} = -E\chi x$

($\chi = \frac{1}{R}$ là độ cong cần tìm của trục dầm sau khi uốn, R – bk cong)

Các thành phần US thay vào pt cân bằng ta được $K_x = K_y = K_z = 0$ – tồn tại khi không có lực khối.

ĐKB: + Tại mặt bên $n = 0$, $\Sigma x = \Sigma y = \Sigma z = 0$

+ Tại 2 đáy $z = \pm L$: $l = m = 0$, $n = \pm 1$ $\Rightarrow \Sigma x = \Sigma y = 0$, $\Sigma z = \pm E\frac{x}{R}$.

+ Tại miền $x > 0$ của đáy $z = L$, lực pháp $\Sigma z = -E\frac{x}{R} < 0$

+ Tại miền $x < 0$ của đáy $z = L$, lực pháp $\Sigma z > 0$

$\Rightarrow$ Hợp lực của các lực pháp trên tiết diện bằng không: $\iint_S \sigma_z dxdy = \frac{-E}{R}\iint_S x dxdy = 0$

Momen uốn: $M = \iint_S \sigma_z . x dxdy = -\frac{E}{R}\iint x^2 dxdy = -\frac{EJ_y}{R}$ $\quad M = -EJ_y\chi$

$\Rightarrow$ Độ cong: $\chi = -\frac{M}{EJ_y}$, $J_y = \iint x^2 dxdy$ moment quán tính với trục y, EJ_y là độ cứng khi uốn.

ĐL Hook: $\varepsilon_z = \frac{\sigma_z}{E} = -\frac{x}{R} = -\chi x$, $\varepsilon_x = \varepsilon_y = \frac{-\nu\sigma_z}{E} = \frac{\nu x}{R} = \nu\chi x$, $\varepsilon_{xy} = \varepsilon_{yz} = \varepsilon_{zx} = 0$

Cauchy: $\frac{\partial u}{\partial x} = \frac{\nu x}{R} = \frac{\partial v}{\partial y}$, $\frac{\partial w}{\partial z} = -\frac{x}{R}$, $\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0 = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$

$\Rightarrow u = \frac{z^2}{2R} + \nu\frac{x^2 - y^2}{2R} - C_1 z + C_2 y + C_3$, $v = \frac{\nu xy}{R} + C_4 z - C_2 x + C_5$, $w = -\frac{xz}{R} + C_1 x - C_4 y + C_6$

ĐKB: $x = y = z = 0$ thì $u = v = w = 0 \Rightarrow C_3 = C_5 = C_6 = 0$

$w = 0$ khi $z = 0$ $(\forall x, y) \Rightarrow C_1 = C_4 = 0$.

$\frac{\partial u}{\partial y} = 0$ với $x = y = z = 0 \Rightarrow C_2 = 0$

$\Rightarrow u = \frac{z^2}{2R} + \nu\frac{x^2 - y^2}{2R}$

$v = \frac{\nu xy}{R}$, $w = -\frac{xz}{R}$